ENEE620-23. Home assignment 3. Date due October 31, 11:59pm EDT.

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Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)
• Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
• Justification of solutions is required.
• Each problem is worth 10 points.

Problem 1. Let P be an $n \times n$ irreducible aperiodic matrix.

(a) Show that the smallest cycle (directed, closed path in the graph) of the Markov chain defined by P cannot contain all n states.

(b) Denote by t the length of this cycle and let i be a state in it; let $S_i(m)$ be the set of states accessible from i in exactly m steps. Show that $S_i(m) \subseteq S_i(m+t)$ for all $m \geq 1$.

(c) Show that $\{i\} \subseteq S_i(t) \subseteq S_i(2t) \subseteq \cdots \subseteq S_i(mt) \subseteq \ldots$.

(d) Show that if one of the inclusions in (c) is satisfied with equality, then all the inclusions after that one are also satisfied with equality. Show moreover that at most the first $n-1$ inclusions can be strict, and that $S_i(mt) = S_i((n-1)t)$ for all $m \geq n-1$.

(e) Prove that all states are included in $S_i((n-1)t)$.

(d) Conclude by showing that $(P^m)_{ij} > 0$ for all i, j and all $m \geq (n-1)^2 + 1$.

Problem 2. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. RVs such that $P(X = 1) = P(X = -1) = \frac{1}{2}$ for all X_n. Let $S_0 = 0, S_n = X_1 + \cdots + X_n$ and $M_n = \max\{S_k, 0 \leq k \leq n\}, n \geq 1$.

(a) Do the sequences $(|S_n|)_{n\geq 0}$, $(|M_n|)_{n\geq 0}$, and $(M_n - S_n)_{n\geq 0}$ form first-order, homogeneous Markov chains?

(b) Change the definition of S_n as follows: let $S_0 = x$ and $S_n = x + X_1 + \cdots + X_n$, where x is a nonzero integer number. Will the sequences defined in Part (a) form first-order, homogeneous Markov chains?

Problem 3. Let P be the transition matrix of a Markov chain with the state set $S = \{0, 1, 2, \ldots\}, |S| = \infty$, such that for all $i \in S\backslash\{0\}$, the ith row of P is given as follows:

$$P_{i,i+1} = p_i, \; P_{i,0} = 1 - p_i, \; \text{where } 0 < p_i < 1,$$

and $P_{0,1} = 1$. The other matrix elements are equal to 0.

(a) Prove that the states are recurrent if and only if $\lim_{n\to\infty} \prod_{j=1}^{n} p_j = 0$.

(b) Prove that if all the states are recurrent, then they are positive recurrent if and only if

$$\sum_{k=1}^{\infty} \prod_{j=1}^{k} p_j < \infty.$$

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Problem 4. (a) Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transitions

$$P = \begin{pmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 1-c & 0 & c \end{pmatrix},$$

where $a, b, c \in (0, 1)$. Is this Markov chain irreducible? What can be said about the existence of a stationary distrubution for this Markov chain?

(b) Is it possible that all states of a given Markov chain (not the chain defined in Part (a)) are transient if

(1) the state space is finite;
(2) the state space is countably infinite?

Problem 5. An urn contains W_0 white and B_0 black balls. Every second we remove one ball with no replacement. Let W_k and B_k be the number of white and black balls after $k \geq 1$ seconds. Which of the following sequences form and which do not form a Markov chain:

(a) W_k, (b) $W_k - B_k$, (c) $W_k + B_k$, (d) the pair (W_k, B_k), (e) $\frac{W_k - B_k + 1}{W_k + B_k + 2}$?

Problem 6. A Markov chain $(X_m)_m$ with states $(0, 1, \ldots, n-1)$ is defined by the following transition matrix:

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_{n-1} & p_0 & p_1 & \cdots & p_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{pmatrix}.$$

Assuming that none of the probabilities p_i are 0 or 1, prove that

$$\lim_{m \to \infty} P(X_m = i) = \frac{1}{n}, \quad i = 0, 1, 2, \ldots, n-1.$$