Problem 1

(a) For \( u \in \mathbb{N}_0 \), define 
\[ \text{sgn}(u) = \mathbb{I}_{\{u > 0\}} - \mathbb{I}_{\{u \leq 0\}} = \begin{cases} 1, & u > 0 \\ -1, & u \leq 0 \end{cases} \]

Next, let \( Z_n := \text{sgn}(X_{n-1}) \) be an RV, and note that given \( X_{n-1} \),

(1) \( Z_n \perp X_{\ell}, \quad \ell = 0, 1, \ldots, n-2 \) \ (the RVs are independent)

From the statement of the problem we have

(2) \( X_n = X_{n-1} - Z_n \cdot Y_n, \quad n \geq 1 \)

\[
P(X_n = k \mid X_{n-1} = m_{n-1}, X_{n-2} = m_{n-2}, \ldots, X_0 = m_0) \]
\[
= P(X_{n-1} - Z_n \cdot Y_n = k \mid X_{n-1} = m_{n-1}, X_{n-2} = m_{n-2}, \ldots, X_0 = m_0) \]
\[
= P(Z_n = \text{sgn}(m_{n-1}))(k - m_{n-1}) \]

This probability depends only on the proof of \( Y \) \ (and the value of \( X_{n-1} \))
and is otherwise independent of \( n \) and of the history \( (X_{n-2}) \).

(b) The calculation depends on the reading of the question:

either we sample \( N \) at each time instant \( n \geq 0 \), and put \( N_n = N + n \),

or we sample it once and put \( N_n = N + n \) based on that value.

Since \( N \perp X_n \) for all \( n \), either way we get a Markov chain.

Under the first choice, consecutive realizations of \( N_n \) are independent, and

\[
P_n(X_n = i, N_n = m_n \mid X_{n-1} = i) = p_{ij} \cdot P(N = m_{n-1} - n) \]

(note the dependence on \( n \))

For the second choice, \( P(N_n = m_n \mid N_{n-1} = m_{n-1}) = \mathbb{I}(m_n - m_{n-1} = 1) \),

and the transition probabilities are

\[
P(X_n = i, N_n = m_n \mid X_{n-1} = i, N_{n-1} = m_{n-1}) = p_{ij} \cdot \mathbb{I}(m_n - m_{n-1} = 1) \]
Problem 2

(a) If \( p = 1 \), all states are positive recurrent (and absorbing).
For \( p < 1 \), all states except 0 are transient, while 0 is pos. rec.

If \( p = 0 \), all states except 0 are transient; 0 is pos. recurrent.
If \( p = 1 \), all states are transient.

Now let \( 0 < p < 1 \). Since \( p_{2k, 2k+1} = 0 \) for all \( k, l \geq 0 \),
even-numbered states form a closed comm. class. Since
\[
P_{00}^\infty = p^{n-1}(1-p) \sim \text{Geom}(1-p),
\]

\[
E(0 \rightarrow 0) = \frac{1}{1-p} < \infty
\]

state 0 is positive recurrent, and thus, so are all even states.
The odd-numbered states are transient.

(b) The transitions are governed by the graph

With probability 1 after sufficiently many steps we move from
\( \tau - m \rightarrow \tau - m - 1 \); assuming that \( X_k = \tau - m \), we have
\[
P\left( \bigcup_{n \geq 1} X_{k+n} = \tau - m - 1 \right) = \sum_{n \geq 0} \left( \frac{m}{\tau - c} \right)^n (1 - \frac{m}{\tau - c}) = 1
\]

We conclude that states 1, 2, ..., \( \tau \) are transient, and 0 is pos. recurrent.
Problem 3.

The chain is formed of a single communicating class. The probability of return \( f_{00} \) is

\[
f_{ii} = \sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} p_i = 1 \quad \text{(note that) \quad f_{ii}(n) = p_{ii}^{(n)} \quad \text{first return in } n \text{ steps) return in } n \text{ steps}
\]

so the states are recurrent.

\[
\mu_i = \mathbb{E} \left( \text{return time to } i \right) = \sum_{n=1}^{\infty} n f_{ii}(n) = \sum_{n=1}^{\infty} n p_i
\]

The chain is positive recurrent iff this sum is finite.

Let us find the stationary distribution:

\[
\begin{align*}
p_i, \pi_i + \pi_k &= \pi_i, \quad i \geq 1 \\
p_i, \pi_i + \pi_k &= \pi_i, \quad \pi_k = \pi_i (1 - p_i) \\
p_i, \pi_2 + \pi_3 &= \pi_2, \quad \pi_3 = \pi_i (1 - p_i) - \pi_1 p_2 \\
p_i, \pi_k &= \pi_i (1 - p_i, \ldots, p_{k-1}) = \pi_i \sum_{i=k}^{\infty} p_i, \quad k \geq 2
\end{align*}
\]

Summing these equations on \( k = 2, 3, \ldots \), we obtain

\[
\begin{align*}
1 - \pi_i &= \pi_i \sum_{k=2}^{\infty} \sum_{l=k}^{\infty} p_l = \pi_i \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} (k-1) p_k \\
1 &= \pi_i (1 + \sum_{k=2}^{\infty} k p_k) = \pi_i \left( \sum_{k=1}^{\infty} p_k + \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} p_k \right) = \pi_i \sum_{k=1}^{\infty} k p_k
\end{align*}
\]

\[\therefore \pi_i = \frac{1}{\sum_{i=1}^{\infty} i p_i}, \quad \quad \pi_k = \frac{\sum_{i=k}^{\infty} i p_i}{\sum_{i=1}^{\infty} i p_i}, \quad k = 1, 2, \ldots\]
Problem 4

(a) 

![Diagram](image)

The absorbing state is clearly state 1.

Let \( a_m \) be expected time to 1 starting in \( m = 1, 2, 3 \).

We have \( a_1 = 0 \),

\[
    a_2 = \frac{1}{2} (a_2 + 1) + \frac{1}{2} (a_1 + 1), \quad \text{so} \quad a_2 = 2
\]

and

\[
    a_3 = \frac{1}{2} (a_3 + 1) + \frac{1}{4} (a_3 + 1) + \frac{1}{4} (a_3 + 1) = \frac{1}{4} a_3 + \frac{3}{2} + \frac{1}{4} + \frac{1}{4}
\]

implying that \( a_3 = \frac{8}{3} \).

(b) Consider the subset of the chain leading from \( O \) to \( HHT \):

![Diagram](image) [all arrows are \( \frac{1}{2} \)]

where \( O \) means that no symbols out of the needed sequence \( HHT \) have been accumulated.

We have

\[
    a_0 = \frac{1}{2} (a_0 + 1) + \frac{1}{2} (a_{HH} + 1) = \frac{1}{2} a_0 + \frac{1}{2} a_0 + \frac{1}{2}
\]

\[
    a_H = \frac{1}{2} (a_0 + 1) + \frac{1}{2} (a_{HH} + 1) = \frac{1}{2} a_0 + \frac{1}{2} a_{HH} + \frac{1}{2}
\]

\[
    a_{HH} = \frac{1}{2} (a_{HH} + 1) + \frac{1}{2}
\]

and solving for \( a_0 \) we obtain \( a_0 = 8 \).

Similarly, for \( HHT \) we have

![Diagram](image)
\[ a_0 = \frac{1}{2} (a_0 + 1) + \frac{1}{2} (a_H + 1) \]
\[ a_H = \frac{1}{2} (a_H + 1) + \frac{1}{2} (a_{HT} + 1) = \frac{1}{2} a_H + \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} a_H + 1 \right) = \frac{3}{4} a_H + \frac{3}{2} \]
\[ a_{HT} = \frac{1}{2} (a_H + 1) + \frac{1}{2} \left( \frac{1}{2} a_H + 1 \right) \]
\[ \therefore a_H = 6 \]
\[ a_0 = \frac{1}{2} a_0 + \frac{1}{2} + \frac{1}{2} = \frac{1}{2} a_0 + \frac{1}{2} \]
\[ \therefore a_0 = 8 \]

**Problems.**

(a) The process is periodic, so it's **not ergodic**

(b) The process is formed of 2 different comm. classes, so it's **not ergodic**

(c) State 1 is transient, so the limiting distribution is \((\pi_0 = 0, \pi_1 = 0)\)
   The process is **ergodic**

(d) \[
\begin{array}{ccc}
  & \frac{1}{2} & 1 \\
 0 & \frac{1}{2} & \end{array}
\]
   Irreducible + aperiodic, so it is **ergodic**.
   The limiting distribution is \((\frac{2}{3}, \frac{1}{3})\)

(e) \[
\begin{array}{ccc}
  & \frac{1}{2} & 1 \\
 0 & \frac{1}{2} & \end{array}
\]
   State 0 is transient, so the limiting distribution is \((0,1)\)
   The process is **ergodic**

(f) \[
\begin{array}{ccc}
  & \frac{1}{2} & 1 \\
 0 & \frac{1}{2} & \end{array}
\]
   Irreducible + aperiodic, so it is **ergodic** with
   limiting distribution \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\)

(g) States 1, 2, 3 are transient, and state 0 is absorbing.
   The limiting distribution \((1,0,0,0)\) is also stationary,
   so the process is **ergodic**.