ENEE620-23. Home assignment 2. Date due September 30, 11:59pm EDT.

Instructor: A. Barg

Please submit your work as a single PDF file to ELMS (under the "Assignments" tab)

- Papers submitted as multiple pictures of individual pages are difficult for grading and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. Let $(X_n)_n$ be a sequence of independent RVs such that $X_1 = 0$, and for $n \geq 2$,

$$P(X_n = n) = P(X_n = -n) = f(n); P(X_n = 0) = 1 - 2f(n),$$

where $f(n), 0 < f(n) < 1$ is some non-constant function of n (such as $f(n) = \frac{1}{n}$ or similar). Further, let $S_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the empirical mean of the sequence.

(a) Give an example of $f(n)$ such that $S_n \xrightarrow{p} 0$ (converges to 0 in probability).

(b) Give an example of $f(n)$ such that S_n converges almost surely.

(c) Give an example of $f(n)$ such that S_n converges in probability and does not converge almost surely.

Problem 2. In our discussion of the monotone convergence theorem we proved that under some assumptions on RVs $(X_n)_n$, it is true that $E\sum_{n=1}^\infty X_n = \sum_{n=1}^\infty EX_n$. Give an example of an infinite sequence of RVs $(X_n)_n$ for which this equality fails to hold (obviously, they do not satisfy the mentioned assumptions).

Problem 3. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed (i.i.d.) RVs such that $E[X_1^{-1}]$ exists, and let $S_m = X_1 + \cdots + X_m$. Prove that if $m \leq n$, then $E[\frac{S_m}{S_n}] = \frac{m}{n}$.

Problem 4. Let X_1, X_2, X_3 be i.i.d. RVs with pdf $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ o/w, i.e., $X_i \sim \text{Exp}(1)$ for all $i = 1, 2, 3$. Show that the RVs

$$Y_1 = \frac{X_1}{X_1 + X_2}, \; Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \text{ and } Y_3 = X_1 + X_2 + X_3$$

are independent.

Problem 5. Let $\Omega = \mathbb{R}$ and $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra. Fix a number $x \in \mathbb{R}$ and for all $A \in \mathcal{B}(\mathbb{R})$, define

$$\delta_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}.$$

(a) Show that δ_x is a measure, and that $(\Omega, \mathcal{B}(\Omega), \delta_x)$ is a probability space. This measure is in fact called the delta function at x (an impulse, as an engineer would say).

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Show that $\int_{\mathbb{R}} f d\delta_x = f(x)$.

(c) For $A \in \mathcal{B}(\mathbb{R})$ define $\mu(A) = \sum_{n=1}^\infty \delta_n(A)$ to be the couning measure (a.k.a. an impulse train). $\mu(A)$ is simply the number of natural numbers in A, which can of course be finite or infinite. For a nonnegative measurable function f show that $\int_{\mathbb{R}} f d\mu = \sum_{n=1}^\infty f(n)$. This shows that a sum is simply an integral against a counting measure.

SEE NEXT PAGE

Problem 6. (a) Given a CDF F, we denote $F(x-) := \lim_{y \uparrow x} F(y)$ and similarly for $F(x+)$. From the definition of F we know that $F(x) = F(x+)$ for all $x \in \mathbb{R}$. If $F(x-) \neq F(x)$, then $F(x)$ has a discontinuity at x. Your task is to show that the number of points at which $F(x)$ has a discontinuity is at most countably infinite. One option is to use the relation $\{x : F(x) \neq F(x-)\} = \cup_{n=1}^{\infty} \{x : F(x) - F(x-)\} \geq \frac{1}{n}\}$.

(b) Given a CDF F and a number $a \geq 0$, show that

$$\int_{-\infty}^{\infty} (F(x+a) - F(x))dx = a.$$