Problem 2

We have \( |Y_n| = |Y_n - EY_n + EY_n| \leq |Y_n - EY_n| + \frac{1}{n} \)

\( \forall \varepsilon > 0 \)

\[
P(|X_n - x| > \varepsilon) = P(|Y_n| > \varepsilon)
\leq P(|Y_n - EY_n| + \frac{1}{n} > \varepsilon)
= P(|Y_n - EY_n| > \varepsilon - \frac{1}{n})
\leq \frac{\text{Var}(Y_n)}{(\varepsilon - \frac{1}{n})^2}
= \frac{\sigma^2}{n(\varepsilon - \frac{1}{n})^2} \to 0
\]

This implies that \( X_n \xrightarrow{p} X \).

Problem 4

Since \( f(x) = \frac{x}{1+x} \) \( \uparrow \) for \( x > 0 \), we have

\[
P(|X_n| > \varepsilon) = P\left(\frac{|X_n|}{1+|X_n|} > \frac{\varepsilon}{1+\varepsilon}\right) \leq \frac{\varepsilon}{1+\varepsilon} E\left(\frac{|X_n|}{1+|X_n|}\right)
\]

This shows the "if" part.

To prove "only if," suppose that \( X_n \xrightarrow{p} 0 \), then

\[
E\left(\frac{|X_n|}{1+|X_n|}\right) \leq \frac{\varepsilon}{1+\varepsilon} P(|X_n| \leq \varepsilon) + 1 - P(|X_n| > \varepsilon) \to \frac{\varepsilon}{1+\varepsilon}
\]

\( \downarrow n \to \infty \)

which gives the desired result since \( \varepsilon \) is arbitrarily small.

Problem 1

\[
E(X_n^2 - x^2) \leq E(|X_n^2 - x^2|) \leq E((X_n - x)^2) + 2E|X_n|E|X_n - x| \leq E((X_n - x)^2) + 2E|X_n| \sqrt{E(X_n^2)} \rightarrow 0.
\]

\( \uparrow \) Cauchy-Schwarz
Problem 3

Per [H] p. 29

\[ f_{U_1^n}(u^n_i) = \frac{1}{\det J} f_{T^n_i}(t^n_i) \]

Let us compute the Jacobian:

\[ \frac{\partial u}{\partial t} = \begin{bmatrix}
1 & 1 & 0 \\
-1 & -1 & \vdots \\
0 & \vdots & -1
\end{bmatrix} \]

so \( \det J = 1 \).

Further, \( T_1 = u_1, T_2 = u_1 + u_2, \ldots, T_m = u_1 + \ldots + u_m \), and thus

\[ f_{U_1^n}(u^n_i) = \begin{cases}
\lambda^n e^{-\lambda(u_1 + \ldots + u_m)}, & u_1 > 0, \ldots, u_m > 0 \\
0, & \text{otherwise}
\end{cases} \]
Problem 5(c)

(i) \( \mathbb{E} X_n = 0 \)

(ii) \(|S_n|\) is the smallest if the realization is \(-3, -3^2, \ldots, -3^n, 3, \ldots, 3^n\), i.e.,
\[ |S_n| \geq \frac{3^n}{2} + \frac{3}{2^n} \text{ w.p. 1; } \lim_{n \to \infty} \frac{R_n}{n} = \infty \]

(iii) For all \( \varepsilon > 0 \), \( P\left( \frac{|S_n|}{n} > \varepsilon \right) \to 0 \)

Note that, at the same time, \( S_n \xrightarrow{a.s.} 0 \) by SLLN

(a) Consider a sequence of RVs \( Y_n \) defined on \( \Omega = [0, 1] \) and such that
\[ Y_n(w) = \begin{cases} n & 0 \leq w \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \]

Then \( P\left( \left| \frac{Y_n}{n} \right| > \varepsilon \right) \leq \frac{1}{n} \to 0 \), i.e., \( \left| \frac{Y_n}{n} \right| \xrightarrow{P} 0 \).

Now take \( Z_n, 0 = Y_n, Z_{n,1} = Y_n + \frac{1}{n}, Z_{n,2} = Y_n + \frac{2}{n}, \ldots, Z_{n,n-1} = Y_n + \frac{n-1}{n} \). The sequence
\[ \frac{Z_{n,j}}{n} \xrightarrow{P} 0 \]

at the same time, \( \frac{Z_{n,j}}{n} \xrightarrow{a.s.} 0 \)

However, \( \frac{Z_{n,j}}{n^2} \xrightarrow{a.s.} 0 \)

Because as \( n \) increases, the sequence \( \frac{1}{n^2} Z_{n,j}(w) \to 0 \) with probability 1.
Prob. 5(c).

Let $\delta \to 0$. Since $X_n \xrightarrow{P} X$, $P(|X - X_n| > \delta) < \delta$, so for a sufficiently large $n$,

$$P(|X - X_n| > \delta) = P((X - X_n > \delta) \cup (X - X_n < -\delta)) = P(X - X_n > \delta) + P(X - X_n < -\delta) < \delta$$

\[\therefore P(X - X_n < \delta) > 1 - \delta \text{ and } P(X - X_n > -\delta) > 1 - \delta\]

Now choose $c$ such that $P(X < c) > 2\varepsilon$ and $P(X > c + \varepsilon) > 2\varepsilon$ which is possible unless $X = \text{const a.s.}$

Then

$$P((X < c) \cap (X_n < X + \delta)) > 2\varepsilon - \delta$$

$$P((X > c + \varepsilon) \cap (X_n > X - \delta)) > 2\varepsilon - \delta$$

From the first of these, for sufficiently large $n$,

$$P(X_n < c + \delta) > P((X < c) \cap (X_n < X + \delta)) > 2\varepsilon - \delta,$$

and since $\delta$ is arbitrarily small,

$$P(X_n < c) > \varepsilon$$

Similarly $P(X_n > c + \varepsilon) > \varepsilon$, both for sufficiently large $n$.

Now let $n, m$ be large enough so that $P(|X_n - X_m| > \varepsilon) < \varepsilon^3$.

Thus

$$\varepsilon^3 > P(|X_n - X_m| > \varepsilon) \geq P(X_n < c, X_m > c + \varepsilon) \geq P(X_n < c)(X_m > c + \varepsilon) \geq \varepsilon^2.$$

The obtained contradiction concludes the proof.
Problem 5 (d) We have \( E X_n = 0 \); \( \text{Var} (X_n) = E X_n^2 = 2 \cdot n^2 \cdot \frac{1}{2n \log n} = \frac{n}{\log n}, n \geq 2 \)

\( E S_n = 0 \); \( \text{Var} (S_n) = \sum_{i=2}^{n} \frac{i}{\log i} \)

To prove that \( \frac{S_n}{n} \to 0 \), we can use Chebyshev's inequality. Let us show that \( \frac{\text{Var}(S_n)}{n^2} \to 0 \); this will follow from the next line

\[
\sum_{i=2}^{n} \frac{i}{\log i} = \sum_{i=2}^{\sqrt{n}} \frac{i}{\log i} + \sum_{i=\sqrt{n}+1}^{n} \frac{i}{\log i} < \sum_{i=2}^{\sqrt{n}} \frac{i}{\log i} + \frac{1}{\log \sqrt{n}} \sum_{i=2}^{\sqrt{n}} i < \frac{\sqrt{n} \log (\sqrt{n})}{2} + \frac{(\text{const}) n^2}{\log n}
\]

Then \( \frac{\text{Var}(S_n)}{n^2} \to 0 \), proving that \( \frac{S_n}{n} \to 0 \)

At the same time, let \( A_n = \{ \omega : |X_n(\omega)| > n \} \), then

\[
\sum_{n=1}^{\infty} P(A_n) = \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty
\]

so \( P (A_n \text{ i.o.}) = 1 \), i.e., \( |X_n| > 1 \) i.o. At the same time, if \( \frac{S_n}{n} \to 0 \), then with prob. 1, the event \( |\frac{S_n}{n}| < \varepsilon \) occurs infinitely often.

Writing

\[
\left| \frac{X_n}{n} \right| = \left| \frac{S_n - S_{n-1}}{n} \right| \leq \frac{|S_n|}{n} + \frac{|S_{n-1}|}{n} \leq \frac{|S_n|}{n} + \frac{|S_{n-1}|}{n-1}
\]

Thus, if \( |\frac{S_n}{n}| > \varepsilon \) occurs only finitely many times, it is impossible that \( |\frac{X_n}{n}| > 1 \) i.o. This contradiction refutes the assumption \( \frac{S_n}{n} \to 0 \) made above.

Remark: Does this result not contradict SLLN? No, because for SLLN for the sum of independent, not identically distributed RV's it is needed to hold, \( \text{Var}(X_i) \) have to be uniformly bounded for all \( i = 1, 2, \ldots \). In this problem, \( \text{Var}(X_i) \to \infty \), so the sufficient condition is not satisfied.