Problem 1. Let $\Omega$ be the set of all infinite sequences of tosses of a fair coin. Let $A_n$ denote the event that the $n$th toss is 1.

(a) Consider the event $E = (\limsup_n A_n) \cap (\liminf_n A_n)$. Is it empty? If yes, give a proof; if no, give an example of $\omega$ that is contained in $E$, with a justification.

(b) What is the probability $P(A_{2n} \text{ i.o.})$?

(c) Consider the event $D$ formed of all the sequences $\omega$ with all odd entries equal to 1. What is the probability of $D$?

(a) $\liminf_n A_n = \bigcap_{n \geq 1} \bigcup_{m \geq n} A_m$ means that for some $n_0$, $\omega \in \bigcap_{m \geq n_0} A_m$,
i.e., $\omega_1 = 1$ for all $m \geq n_0$. In other words, $\omega$ contains only finitely many 0’s.

$\limsup_n A_n = \bigcup_{n \geq 1} \bigcap_{m \geq n} A_m$ means that $\omega$ contains infinitely many 1’s.

Thus, $\liminf_n A_n \subseteq \limsup_n A_n$, and any $\omega = \overbrace{\text{111...}}^{n_0}$ is contained in both sets.

(b) $P(A_{2n}) = \frac{1}{2}$ for all $n$, so by the Borel-Cantelli Lemma 2, $P(A_{2n} \text{ i.o.}) = 1$

(c) Let $D_m = \{ \omega : \omega_{2i+1} = 1 \text{ for all } i = 0, 1, \ldots, m \}, \quad m = 0, 1, \ldots$

We have $D = \bigcap_{m = 0}^{\infty} D_m$ and $D_0 \supset D_1 \supset \ldots$

$P(D) = P(\bigcap_{m = 0}^{\infty} D_m) = \lim_{m \to \infty} P(D_m) = \lim_{m \to \infty} \frac{1}{2^{m+1}} = 0$
Problem 2. Let \( \Omega \) be as in Problem 1, and let \( \mathcal{F}_n \) be a collection of subsets of \( \Omega \) that depend only on the result of the first \( n \) tosses. In other words, \( A \subseteq \mathcal{F}_n \) if and only if there is a subset \( A^{(n)} \subseteq \{0,1\}^n \) such that \( A = \{ \omega \in \Omega \mid (\omega_1, \omega_2, \ldots, \omega_n) \in A^{(n)} \} \). For instance, \( A \) is the set of \( \omega \)'s with exactly 3 ones among the first 10 tosses, etc.

(a) Prove that \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) for all \( n \).
(b) Prove that the set \( \mathcal{F}_0 := \bigcup_{i \in \mathbb{N}} \mathcal{F}_i \) is an algebra.
(c) Prove that \( \mathcal{F}_0 \) is not a \( \sigma \)-algebra.
(d) Consider the event
\[
T := \{ \omega \in \Omega : \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \omega_i}{n} = \frac{1}{2} \}.
\]
Is \( T \) contained in \( \mathcal{F}_0 \)?

Problem 3. Let \( \Omega \) be an arbitrary set.

(a) Let \( i_1, \ldots, i_n \in \{0,1\}^n \) be a fixed sequence of \( n \) bits and let
\[
B^{(n)}_{i_j} = \{ \omega : \omega_j = i_j \text{, } j = 1, \ldots, n \}
\]
The set \( \mathcal{F}_n \) is formed by all finite unions of the events \( B^{(n)}_{i_j} \) since any event \( A \subset A^{(n)} \) can be written as such a union. Clearly, \( B^{(n)}_{i_j} \subset B^{(n+1)}_{j} \) for some \( j \), and this inclusion is strict, so \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \). Note that \( |\mathcal{F}_n| = 2^n \).

(b) Let \( A, B \in \mathcal{F}_0 = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i \). Then there is \( n_A \) s.t.
\[
A \in \mathcal{F}_{n_A} \text{ and } A \notin \mathcal{F}_n \text{ for all } n > n_A.
\]
Define \( n_B \) analogously, and note that \( A \cup B \in \mathcal{F}_{\max(n_A, n_B)} \subset \mathcal{F}_0 \).

(c) At the same time, let
\[
C = \{ \omega \mid \omega_1 = \ldots = \omega_{n-1} = 0, \omega_n = 1 \}, \quad n \geq 1
\]
Then \( C \in \mathcal{F}_n \), but \( C \notin \mathcal{F}_0 \) because \( C \) cannot be decided by any finite number of outcomes.

(d) \( T \) is the set of "normal numbers",
\[
T = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ \omega : \left| \frac{1}{n} \sum_{i=n+1}^{n+m} \omega_i - \frac{1}{k} \right| < \frac{1}{k} \right\}
\]
Thus \( T \) requires countable intersections and unions, so \( T \notin \mathcal{F}_0 \).
Problem 3. Let $\Omega$ be an arbitrary set.
(a) Is the collection $\mathcal{F}_1$ consisting of all finite subsets of $\Omega$ an algebra?

(b) Let $\mathcal{F}_2$ consist of all finite subsets of $\Omega$ and all subsets of $\Omega$ having a finite complement. Is $\mathcal{F}_2$ an algebra?

(c) Is $\mathcal{F}_2$ a $\sigma$-algebra?

(d) Let $\mathcal{F}_3$ consist of all countable subsets of $\Omega$ and all subsets of $\Omega$ having a countable complement. Is $\mathcal{F}_3$ a $\sigma$-algebra?

(a) $\mathcal{F}_1$ is an algebra if $|\Omega| < \infty$ and isn’t otherwise (the complement of a finite set is infinite).

(b) First, $\Omega \in \mathcal{F}_2$ because $\Omega^c = \emptyset$ is finite.

Next, if $A \in \mathcal{F}_2$, then also $A^c \in \mathcal{F}_2$.

Finally, we show that $A, B \in \mathcal{F}_2$ implies that $A \cup B \in \mathcal{F}_2$.

1. $|A| < \infty$, $|B| < \infty$. Then $A^c \in \mathcal{F}_2$, $B^c \in \mathcal{F}_2$, $A \cup B \in \mathcal{F}_2$.

2. $|A| = \infty$, $|B| < \infty$. Then $|A^c| < \infty$, so $A^c \in \mathcal{F}_2$; also $B \in \mathcal{F}_2$.

   $$(A \cup B)^c = A^c \cap B^c \subset B^c$$

   Since $B^c$ is finite, so is $(A \cup B)^c$, and thus $A \cup B \in \mathcal{F}_2$.

3. $|A| = \infty$, $|B| = \infty$, and $|A^c| < \infty$, $|B^c| < \infty$.

   $$(A \cup B)^c = A^c \cap B^c \text{ is finite since both } |A^c| < \infty, |B^c| < \infty$$

   $\therefore A \cup B \in \mathcal{F}_2$.

(c) $\mathcal{F}_2$ is generally not a $\sigma$-algebra. For instance, take

$\Omega = \mathbb{R}$, $A_n = \{n\}$ for all $n \geq 1$. Then $\bigcup_{n \geq 1} A_n \notin \mathcal{F}_2$ because neither $\bigcup_{n \geq 1} A_n$ is finite nor $(\bigcup_{n \geq 1} A_n)^c$ is finite.

(d) If countable means countably infinite, then $\mathcal{F}_3$ is not a $\sigma$-algebra. For instance, if $\Omega = \mathbb{R}$, then neither $\Omega$ nor $\Omega^c = \emptyset$ is countably infinite.
If countable is understood as at most countably infinite, then $F_3$ is a $\sigma$-algebra. Let us assume that $\Omega$ is $\mathbb{R}$. Then it is clear that $\Omega \in F_3$ and if $A \in F_3$ then $A^c \in F_3$. We are left to show that if $(A_n)_n$ is a collection of subsets s.t. $A_n \in F_3$ for all $n$, then $\bigcup_{n=1} \in F_3$. The main observation is that a countable union of countably infinite sets is itself countable.

1. Suppose all $A_n$ are countable, then $\bigcup_{n=1} \in F_3$

2. Suppose all $A_n$ are such that $A_n^c$ are countable, then

$$\bigcup_{n} A_n = (\bigcap_{n} A_n)^c = (\text{countable})^c \in F_3$$

3. Suppose that $N = \{n_1, n_2, \ldots\}$ is the subset of indices s.t. $A_{n_i}^c$ is countable for each $i$, and all the other subsets $A_n, n \notin N$ are themselves countable. Then

$$\bigcup_{n \in N} A_n = \bigcup_{n \in N} A_n \cup \bigcup_{n \notin N} (\bigcap_{n \in N} A_n)^c \cup \bigcup_{n \notin N} A_n$$

$$= B \cup \bigcup_{n \notin N} A_n$$

where $B$ is such that its complement is countable.

Observe that

$$(B \cup \bigcup_{n \notin N} A_n)^c = B^c \cap (\bigcup_{n \notin N} A_n)^c \subset B^c,$$

which proves our claim.

Finally, if $|\Omega| > |\mathbb{R}|$, then $F_3$ may fail to be a $\sigma$-algebra.
Problem 4. Consider a sequence of independent identically distributed geometric RVs \( (X_n)_{n} \) with probability of success \( p \) and probability of failure \( q = 1 - p \). Thus, we have \( P(X_n = m) = pq^{m-1} \) for all \( m \geq 1 \) and all \( n \).

(a) What is the probability \( P(X_n \geq t) \), where \( t \in \mathbb{N} \)? What is \( EX_n \)?

(b) Find the value of \( a \) such that \( P(X_n \geq (1 + \epsilon)a \ln n) \) i.o. = 1 for all \( \epsilon > 0 \).

(c) For the value of \( a \) you found in part (b), show that \( P(X_n \geq (1 + \epsilon)a \ln n) \) i.o. = 0 for all \( \epsilon > 0 \).

(d) Using the results in parts (b) and (c), what is the probability of the event \( \left\{ \omega : \limsup_{n \to \infty} \frac{X_n}{a \ln n} = 1 \right\} \)?

(e) Is there a contradiction between (d) and the value of \( EX_n \)? Give a yes/no answer and a justification.

(a) \[ P(X_n = t) = \frac{t-1}{q} \]

\[ P(X_n \geq t) = \sum_{i=t}^{\infty} \frac{1}{q} = \frac{1}{1-q} = q^{-1} \]

\[ EX_n = \sum_{t=1}^{\infty} tP(X_n \geq t) = \sum_{t=1}^{\infty} q^{t-1} = \frac{q}{1-q} \]

(b) \[ P(X_n \geq (1-\epsilon)a \ln n) = \frac{q^{-(1-\epsilon)}a \ln n}{\log q} = \frac{(1-\epsilon)a \ln n}{\log q} \] \[ \leq n^{(1-\epsilon)a \ln q} = \infty \text{ for } (1-\epsilon) a \ln q > 1 \quad \text{(Borel-Cantelli)} \]

The largest \( a \) for which this inequality holds is \( a = \frac{1}{\ln q} \)

(c) With \( a = \frac{1}{\ln q} \)

\[ P(X_n \geq ((1+\epsilon)a \ln n)) = \frac{q^{-(1+\epsilon)}a \ln n}{\log q} < \infty \]

and \( P(X_n \geq (1+\epsilon) a \ln n \text{ i.o.}) = 0 \) \quad \text{(Borel-Cantelli)}

(d) Parts (b) and (c) hold for arbitrarily small \( \epsilon > 0 \).

\[ : \quad P(\omega : \limsup_{n \to \infty} \frac{X_n}{a \ln n} = 1) = 1 \]

(e) Given that the expected value of \( X_n = \frac{1}{p} \), it looks strange that \( X_n \) hits \( a \ln n \) i.o. with prob 1. This is due to the fact that the geometric distribution declines slowly (heavy tails), allowing values on the order of \( \ln n \) i.o.
Problem 5. (a) Show that a random variable $X$ has a continuous distribution if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

(b) Show that if $X$ and $Y$ are RVs on a probability space $(\Omega, \mathcal{F}, P)$, then the set \( \{ \omega : X(\omega) = Y(\omega) \} \in \mathcal{F} \).

(c) Give an example of two different RVs $X$ and $Y$ whose CDFs $F_X$ and $F_Y$ coincide.

(a) If for some $x_0$, $P(X = x_0) > 0$, then
\[ \lim_{x \downarrow x_0} F_X(x) \neq \lim_{x \uparrow x_0} F_X(x), \]
so $F_X$ is not continuous at $x_0$.

Conversely, if $F_X$ is continuous, then $\forall x_0 \in \mathbb{R}$,
\[ \lim_{x \uparrow x_0} F_X(x) = F_X(x_0) \quad \text{and} \quad \lim_{x \downarrow x_0} F_X(x) = F_X(x_0) \]
\[ \Leftrightarrow \forall \delta > 0 \exists \varepsilon_1, \text{ s.t. } F_X(x_0) - F_X(x) \leq \frac{\delta}{2} \quad \text{once } x - x_0 < \varepsilon \ (x_0 > x) \]
\[ \text{and} \quad \exists \varepsilon_2, \text{ s.t. } F_X(x) - F_X(x_0) \leq \frac{\delta}{2} \quad \text{once } x - x_0 < \varepsilon_2 \ (x_0 < x) \]
Thus
\[ F_X(x_0) - F_X(x) < \delta \]
Since $\delta$ is arbitrarily small, and $F_X(x_0) = F_X(x_0) = F_X(x)$,
this implies that
\[ P(X = x_0) = \lim_{\varepsilon \to 0} (F_X(x_0 + \varepsilon) - F_X(x_0 - \varepsilon)) = 0 \]

(c) Let $(\Omega, \mathcal{B}(0,1), \lambda)$ be the standard prob. space.

The RVs $X_1$ and $X_2$ are different, but have the same CDF.
(b) \( \{ \omega : X(\omega) = Y(\omega) \} = \{ \omega : X(\omega) \leq Y(\omega) \} \cap \{ \omega : Y(\omega) \leq X(\omega) \} \)

Now, \( Z = X - Y \) is an RV, and thus \( \{ \omega : Z(\omega) \leq 0 \} \in \mathcal{F} \)

and also \( -Z \) is an RV, and \( \{ \omega : Z(\omega) \geq 0 \} \in \mathcal{F} \).

Or, we can say that
\[
\{ X \leq Y \} = \bigcup_{x \in \mathbb{Q}} \left( \{ X \leq x \} \cap \{ Y \geq x \} \right)
\]

which is a countable union of events, so itself an event.

Since rational numbers are dense in \( \mathbb{R} \), we conclude that
\( \{ X \leq Y \} \) is an event.