ENEE620-23. Home assignment 1. Date due September 18, 11:59pm EDT.

Instructor: A. Barg

Please submit your work as a single PDF file to ELMS (under the "Assignments" tab)
- Papers submitted as multiple pictures of individual pages are difficult for grading and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. Let Ω be the set of all infinite sequences of tosses of a fair coin, let A_n denote the event that the nth toss is 1.

(a) Consider the event $E = (\limsup_n A_n) \cap (\liminf_n A_n)$. Is it empty? If yes, give a proof; if no, give an example of ω that is contained in E, with a justification.

(b) What is the probability $P(A_{2n} \text{ i.o.})$?

(c) Consider the event D formed of all the sequences ω with all odd entries equal to 1. What is the probability of D?

Problem 2. Let Ω be as in Problem 1, and let $\mathcal{F}_n$ be a collection of subsets of Ω that depend only on the result of the first n tosses. In other words, $A \in \mathcal{F}_n$ if and only if there is a subset $A^{(n)} \subset \{0,1\}^n$ such that $A = \{\omega \in \Omega \mid (\omega_1, \omega_2, \dots, \omega_n) \in A^{(n)}\}$. For instance, A is the set of ω's with exactly 3 ones among the first 10 tosses, etc.

(a) Prove that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n.

(b) Prove that the set $\mathcal{F}_0 := \cup_{i \in \mathbb{N}} \mathcal{F}_i$ is an algebra.

(c) Prove that $\mathcal{F}_0$ is not a σ-algebra.

(d) Consider the event

$$T := \Big\{\omega \in \Omega : \lim_{n\to\infty} \frac{\sum_{i=1}^n \omega_i}{n} = \frac{1}{2}\Big\}.$$

Is T contained in $\mathcal{F}_0$?

Problem 3. Let Ω be an arbitrary set.

(a) Is the collection $\mathcal{F}_1$ consisting of all finite subsets of Ω an algebra?

(b) Let $\mathcal{F}_2$ consist of all finite subsets of Ω and all subsets of Ω having a finite complement. Is $\mathcal{F}_2$ an algebra?

(c) Is $\mathcal{F}_2$ a σ-algebra?

(d) Let $\mathcal{F}_3$ consist of all countable subsets of Ω and all subsets of Ω having a countable complement. Is $\mathcal{F}_3$ a σ-algebra?

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Problem 4. Consider a sequence of independent identically distributed geometric RVs $(X_n)_n$ with probability of success p and probability of failure $q = 1 - p$. Thus, we have $P(X_n = m) = pq^{m-1}$ for all $m \geq 1$ and all n.

(a) What is the probability $P(X_n \geq t)$, where $t \in \mathbb{N}$? What is EX_n?

(b) Find the largest value a such that $P(\{X_n \geq (1-\epsilon)a \ln n\} \text{ i.o.}) = 1$ for all $\epsilon > 0$.

(c) For the value a you found in part (b), show that $P(\{X_n \geq (1+\epsilon)a \ln n\} \text{ i.o.}) = 0$ for all $\epsilon > 0$.

(d) Using the results in parts (b) and (c), what is the probability of the event

$$\left\{\omega : \limsup_n \frac{X_n}{a \ln n} = 1\right\} ?$$

(e) Is there a contradiction between (d) and the value of EX_n? Give a yes/no answer and a justification.

Problem 5. (a) Show that a random variable X has a continuous distribution if and only if $P(X = x) = 0$ for all $x \in \mathbb{R}$.

(b) Show that if X and Y are RVs on a probability space $(\Omega, \mathcal{F}, P)$, then the set $\{\omega : X(\omega) = Y(\omega)\} \in \mathcal{F}$.

(c) Give an example of two different RVs X and Y whose CDFs F_X and F_Y coincide.