Problem 1. There are \( n \) identical balls and \( m \) labeled boxes. In an experiment, we place each of the balls in a uniformly randomly chosen box independently of the other balls.

1. Construct the sample space of this experiment \( \Omega \).

2. What is the expected number of balls in the first box?

3. Now suppose that \( m = n \) and suppose that the balls are labeled from 1 to \( m \), and so are the boxes. The experiment proceeds as described above. What is the probability that the \( i \)th ball is not in the \( i \)th box, for all \( i = 1, \ldots, n \) ?

Problem 2. We are given a coin such that \( P(\text{Heads}) = p \in (0, 1) \).

(1) Suppose that we toss the coin \( n \) times. Let \( H \) and \( T \) denote the number of heads and tails. Show that \( H \) and \( T \) are dependent RVs.

(2) Now suppose that we toss the coin \( N \) times, where \( N \) is a Poisson RV with expectation \( \lambda \). Again let \( H \) and \( T \) denote the number of heads and tails. Are \( H \) and \( T \) dependent? What are the pmfs of \( H \) and \( T \)?

Problem 3. (a) Let \( \Omega = \{1, 2, 3, 4\} \) and let \( E = \{\{1\}, \{2\}\} \). Describe explicitly the \( \sigma \)-algebra \( \sigma(E) \) generated by \( E \).

(b) Let \( \Omega = \mathbb{N} \) (the set of natural numbers) and let \( \mathcal{F} \) be the set of all subsets \( A \subset \Omega \) such that either \( A \) or \( A^c \) is finite. Define a function \( P : \mathcal{F} \to \{0, 1\} \) by setting \( P(A) = 0 \) if \( |A| < \infty \) and \( P(A) = 1 \) if \( |A^c| < \infty \).

(b1) Is \( \mathcal{F} \) an algebra? Is it a \( \sigma \)-algebra?

(b2) Is \( P \) finitely additive?

(b3) Is \( P \) countably additive on \( \mathcal{F} \), i.e., for any collection of disjoint subsets \( A_n \in \mathcal{F} \), \( n = 1, 2, \ldots \) such that \( \cup_n A_n \in \mathcal{F} \), we have \( P(\cup_n A_n) = \sum_n P(A_n) \)?

Problem 4. We say that an RV \( X \) is defined with respect to a \( \sigma \)-algebra \( \mathcal{F} \), i.e., \( \{\omega : X(\omega) \leq s\} \in \mathcal{F} \) for any real number \( s \).

Let \( X \) and \( Y \) be RVs defined with respect to a \( \sigma \)-algebra \( \mathcal{F} \). Using the definitions of the random variable and \( \sigma \)-algebra, show that

(a) \( X + Y \) is an RV with respect to \( \mathcal{F} \);

(b) \( \max(X, Y) \) is an RV with respect to \( \mathcal{F} \).

Problem 5. Let \( y_n, n = 1, 2, \ldots \) be a number sequence such that \( 0 \leq y_n \leq 1 \) for all \( n \) and

\[
\sum_{n=1}^{\infty} y_n = \infty.
\]
Show that
\[ \prod_{n=1}^{\infty} (1 - y_n) = 0. \]

**Problem 6.** Consider the following experiment. Imagine you have an urn with infinite capacity and an infinite supply of integer-numbered but otherwise identical balls. At 1 minute to noon, you place 10 balls numbered 1 through 10 in the urn and closing your eyes, remove one ball randomly. At 1/2 minute to noon, you place balls 11-20 in the urn and again remove a uniformly chosen random ball from the 19 remaining balls. At 1/4 minute to noon, you place balls 21-30 in the urn and remove a ball randomly from the 28 current balls. And you keep on doing this at 1/8 minute to noon, 1/16 minute to noon and so on for ever and ever. Show that, at noon the urn contains no balls.

**Problem 7.** (Do not submit the solution of this problem as it will not be graded. Please make sure that you understand these results and their proofs.)

You must provide proofs based on definitions. You may use computers if you want, but please do not include computational evidence as a part of your solution.

(a) Prove that for a natural number \( n \neq 1 \),
\[ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2} \]

(b) Let \( H_n = \sum_{i=1}^{n} \frac{1}{i} \). Prove that
\[ H_{2^k} > \frac{k}{2} \]
(hint: partition the sum in \( H_{2^k} \) into groups of size \( 2^i, i = 1, \ldots, k - 1 \))

(c) Prove that
\[ \sum_{n=1}^{\infty} \frac{1}{n} = \infty \]
(the harmonic series diverges).

(d) Let \( s < 1 \) be a real number. Prove that
\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \infty. \]

(e) Let \( s = 1 + t \), where \( t > 0 \) is a real number. Prove that
\[ \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s} < \frac{1}{n^t}. \]

(f) Arguing as in (b)-(c), prove that
\[ \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty \]
i.e., the series on the previous line converges for every \( s > 1 \).

Conclusion: \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) diverges if \( s \leq 1 \) and converges if \( s > 1 \).

(g) Consider again the experiment whose sample space \( \Omega \) is formed of infinite sequences of coin tosses. Let \( p_n \) be the probability that the \( n \)th toss turns Heads.

(g1) Let \( p_n = \frac{1}{n} \). What is the probability that \( \omega \) contains infinitely many \( H \)?

(g2) Let \( p_n = \frac{1}{n^2} \). What is the probability that \( \omega \) contains infinitely many \( H \)?