Problem 1.

1.1 Let $w_i$ be the number of balls placed in $i$th box, $0 \leq w_i \leq n, 1 \leq i \leq m$
Then $\mathcal{N} = \{<w_1, w_2, \ldots, w_m> \text{ s.t. } 0 \leq w_i \leq n, \sum_{i=1}^{m} w_i = n\}$.

1.2 Let $X_i$ be an r.v. determining the number of balls in $i$th box. Due to symmetry
$$E[X_1] = E[X_2] = \ldots = E[X_m].$$
Then $E\left[\sum_{i=1}^{m} X_i\right] = n \rightarrow mE[X_i] = n \rightarrow E[X_i] = \frac{n}{m}.$

Alternatively, let $Z_{ij}$ be a binary r.v. which takes value 1 if ball $j$ is placed
at box $i$, $1 \leq i \leq m, 1 \leq j \leq n$. Then $P(Z_{ij} = 1) = \frac{1}{m}, P(Z_{ij} = 0) = \frac{m-1}{m}.$ Then write
$$X_i = \sum_{j=1}^{n} Z_{ij} \iff E[X_i] = E\left[\sum_{j=1}^{n} Z_{ij}\right] = \sum_{j=1}^{n} E[Z_{ij}] = \frac{n}{m} \cdot \frac{m}{m} = \frac{n}{m}.$$

1.3 Define $A_i$ as the event whereby "The $i$th ball is not placed at $i$th box". Then we're
looking for:
$$P = \prod_{i=1}^{m} P(A_i).$$
Since the placement of each ball is decided independently from that of
others, we can express $P$ as:
$$P = \prod_{i=1}^{m} P(A_i).$$
Then $P(A_i) = P(Z_{ii} = 0) = \frac{m-1}{m}$ as in part 1.2 due to the uniformity of
the measure. Thus, finally:
$$P = \left(\frac{m-1}{m}\right)^m.$$

Notes: 1. How would one proceed if each box were to contain at most 1 ball?
2. In part 1.3 we are in effect looking at random permutations of $n$ elements
$\{1, 2, \ldots, n\}$ such that no $i=1,2,\ldots,n$ is in the $i$th position. Such permutations are called
derangements (wikipedia). Since $\left(\frac{m-1}{m}\right)^n = (1 - \frac{1}{m})^n \rightarrow \frac{1}{e}$ as $m \rightarrow \infty$, the
probability of a derangement is (close to) $\frac{1}{e}$. 

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Problem 2.

\[
\begin{align*}
\Pr_{HT}(h,t) &= \binom{n}{h} p^h (1-p)^{n-h} I_{\{h+t=n\}} , \quad 0 \leq h \leq n \\
\Pr_H(h) &= \binom{n}{h} \ p^h (1-p)^{n-h} \\
\Pr_T(t) &= \binom{n}{t} \ p^t (1-p)^{n-t} \\
\text{One can see } \Pr_{HT}(h,t) &= \Pr_H(h) \Pr_T(t) \quad \text{does not hold. Then } H \text{ and } T \text{ are not independent.} \\
\text{Alternatively, one could give a counter-example where } \boxplus \text{ does not hold. For instance } (h,t) = (n,n).
\end{align*}
\]

2.2 Let us compute \[
\Pr_{HTN}(h,t,n) = \binom{n}{h} \ p^h (1-p)^{n-h} I_{\{h+t=n\}} \frac{n^e^{-\lambda}}{n!} \quad 0 \leq h \leq n, n \geq 0
\]

Now, \[
\Pr_{HT}(h,t) = \sum_{n \geq 0} \binom{n}{h} \binom{n}{t} (1-p)^{n-h} (1-p)^{n-t} \frac{n^e^{-\lambda}}{n!} I_{\{h+t=n\}} =
\]

\[
\binom{h+t}{h} p^h (1-p)^{t} \frac{\lambda^{h+t}}{(h+t)!} I_{\{h+t=n\}} =
\]

\[
\frac{(\lambda p)^h (1-p)^t e^{-\lambda (p+t)}}{h! t!} =
\]

\[
= \text{Poisson}(\lambda p) \times \text{Poisson}(\lambda (1-p))
\]

A simple check shows that these two PMFs correspond to the marginal PMFs \[
\Pr_H(h) \text{ and } \Pr_T(t), \text{ which establishes independence.}
\]

Note: Have this special form and development in mind. You'll see it again soon enough!
Problem 3.

3. a. \( \mathcal{E}(E) = \{ 1 \mathbb{N}, \{ 2, 3, 4 \}, \{ 1, 2 \}, \emptyset \} \)

3. b 1. \( F \) is an algebra:
   \( \checkmark \) 1) \( \mathcal{F} \) is finite \( \Rightarrow \mathcal{F} \in F \) and therefore \( \mathcal{F}^c \in F \)
   \( \checkmark \) 2) \( A \in F \)
   \( \iff \) \( A^c \) is finite \( \Rightarrow A^c \in F \)
   \( \checkmark \) 3) \( A, B \in F \)
   \( \checkmark \) Both finite \( \Rightarrow (A \cup B)^c \) finite \( \Rightarrow A \cup B \in F \)
   \( \checkmark \) Either both \( A^c, B^c \) finite or \( B^c, A^c \) finite:
   W.L.O.G. take \( A^c, B^c \) finite \( \Rightarrow (A \cup B)^c = A^c \cap B^c \subseteq A^c \)

3. b 2. \( F \) is clearly not a \( \sigma \)-algebra, because a countable union of its elements does not necessarily belong to \( F \). For instance take the set of even numbers \( \mathbb{N}_e \).

\[ \mathbb{N}_e = \bigcup_{i=1}^{\infty} A_i \text{ where } A_i = \{ 2i \} \]

Each of \( A_i \)'s belongs to \( \mathbb{N} \) but their union \( \mathbb{N}_e \) does not. Because neither \( \mathbb{N}_e \) nor its complement \( \mathbb{N}_e^c \) are not finite.

3. b 2. \( F \) is finitely additive. To see this, let us take any (finite) collection of disjoint subsets \( A_i \subseteq \mathbb{N} \) i.e. \( \forall i, j: A_i \cap A_j = \emptyset \). Either of the following two cases is possible: 1) All \( A_i \) are finite 2) Exactly one \( A_i \) is infinite. Because otherwise, if for some \( i, j \) we have \( A_i, A_j \) are infinite, we'll have \( A_i \cap A_j \). But \( A_i \) has to be finite; this leads to a contradiction. Now, for either of the two cases we can write:

\[ P( \bigcup_{i=1}^{n} A_i ) = \sum_{i=1}^{n} P(A_i) \]

\[ \text{Case } 1 \rightarrow 0 \]

\[ \text{Case } 2 \rightarrow 0 + 1 = 1 \]

thus finite additivity of \( P \) holds.
Clearly, $P$ is not countably-additive. Take $A_i = \{ i \} \forall i \in \mathbb{N}$. Then, assume towards a contradiction that $P$ is countably-additive. Then write:

$$1 = P(\{ A_i \}) = P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} 0 = 0 \implies P \text{ is not countably-additive.}$$

Problem 4.

4a. First note that $\forall a,b,s \in \mathbb{R}: a+b > s \iff \exists q \in \mathbb{Q} \text{ s.t. } a > q \text{ and } b > s-q$.

The "$\Rightarrow$" direction is straightforward. On the other hand suppose $a+b = t > s$. Then take some $q \in \mathbb{Q}$ s.t. $a > q > a+t-s$. It follows that $b > s-q$. With this we write:

$$\forall s \in \mathbb{R} \exists \{ w : x(w) + y(w) \leq s \} \subseteq \{ w : x(w) + y(w) > s \} = U \{ w : x(w) > q, \ y(w) > s-q \}_{q \in \mathbb{Q}} \cup \left( \{ w : x(w) > q \} \cap \{ w : y(w) > s-q \} \right)_{q \in \mathbb{Q}}$$

But $x(w)$ and $y(w)$ are $F$-measurable random variables. Thus $\{ x(w) \leq q \}$ and $\{ y(w) \leq s-q \}$ both belong to $F$. For all rational $q$ and $s \in \mathbb{R}$. Taking intersections and countable unions of such sets still remains in $F$. Hence,

$$\forall s \in \mathbb{R} \exists \{ w : x(w) + y(w) \leq s \} \in F.$$

Hence, $X+Y$ is an r.v. w.r.t $F$.

4b. Similar to part a write:

$$\forall s \in \mathbb{R} \exists \{ w : \max(x(w), y(w)) \leq s \} \subseteq \{ w : x(w) \leq s \} \cap \{ w : y(w) \leq s \}.$$

Each of the last two sets are in $F$ and so is their intersection. Thus, $\max(X,Y)$ is an r.v. w.r.t $F$. 

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Problem 5.

\[ \forall y \leq 1 : \ c y \leq 1 - y \leq e^{-y} \quad ( \text{Define } f(y) = 1 - y - e^{-y} \text{ and prove this.}) \]

\[ \prod_{n=1}^{\infty} (1 - y_n) = \lim_{k \to \infty} \prod_{n=1}^{k} (1 - y_n) \leq \lim_{k \to \infty} \prod_{n=1}^{k} e^{-y_n} = \lim_{k \to \infty} e^{-\sum_{n=1}^{k} y_n} \]

\[ \text{Now, take } a_k \text{ defined as } a_k = \sum_{n=1}^{k} y_n. \text{ We have } \lim_{k \to \infty} a_k = \infty. \]

Replacing in (\#), the assertion will follow.

Problem 6.

For any ball that is placed in the urn at step \( n \), let \( A_n^{m} \) be the event that the ball is removed at \( m \)-th step. For that ball to remain in the urn at noon, we must have the event \( A_n^{m}\) occurred for all natural \( m > n \). But what is the probability of this happening?

\[ P \left( \bigcap_{m \geq n} A_n^{m} \right) = \prod_{m \geq n} P \left( A_n^{m} \right) = \prod_{m \geq n} \left( \frac{9m}{9m+1} \right) \]

To see this goes to zero, write:

\[ \prod_{m \geq n} \left( \frac{9m+1}{9m} \right) = \prod_{m \geq n} \left( 1 + \frac{1}{9m} \right) \geq \left( 1 + \sum_{m \geq n} \frac{1}{9m} \right) \]

\[ \geq P \left( \bigcap_{m \geq n} A_n^{m} \right) = 0 \quad \text{for all balls added in the } n \text{th step and for all } n \geq 1 \]

Therefore, at noon there will be no balls left in the urn.
Problem 7

(a) \( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2} \)

(b) \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1} + 1} + \cdots + \frac{1}{2k} \)

Let \( n = 2^k \), then

\( H_{2^k} > \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1} + 1} + \cdots + \frac{1}{2^k} > k \cdot \frac{1}{2} \)

(c) Thus \( \lim_{k \to \infty} H_{2^k} = \infty \), i.e. \( \sum_{i=1}^{n} \frac{1}{i} = \infty \)

(d) For \( s < 1 \), \( \frac{1}{n^s} > \frac{1}{n} \) for all \( n \in \mathbb{N} \)

Thus \( \sum_{n=1}^{m} \frac{1}{n^s} > \sum_{n=1}^{m} \frac{1}{n} \); since \( \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n} = \infty \), the same is true for \( \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1}{n^s} \). By definition, this implies the claim.

(e) We have \( \frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s} < n \cdot \frac{1}{n^s} = \frac{1}{n^{s-1}} \)

(f) \( \frac{1}{3^s} + \frac{1}{4^s} < \frac{1}{2^s} \)

\( \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} < \frac{1}{4^s} = (\frac{1}{2^s})^2 \)

\( \vdots \)

\( \frac{1}{(2^k+1)^s} + \cdots + \frac{1}{(2^k)^s} < \frac{1}{(2^k)^s} = \frac{1}{(2^s)^{k-1}} \)

Since \( \frac{1}{2^k} + \frac{1}{(2^k)^2} + \frac{1}{(2^k)^3} + \cdots = \frac{\frac{1}{2^k}}{1 - \frac{1}{2^k}} \), for any \( m > 1 \) we have

\( \sum_{n=1}^{m} \frac{1}{n^s} < 1 + \frac{1}{2^s} + \frac{\frac{1}{2^k}}{1 - \frac{1}{2^k}} \)

Thus, the sums \( \sum_{n=1}^{m} \frac{1}{n^s} \) for increasing \( m \) form an increasing sequence which is bounded above, and therefore has a limit.
(g) (i) \( \sum_{n=1}^{\infty} p_n = \infty \), thus by Borel-Cantelli the required prob. = 1

(2) \( \sum_{n=1}^{\infty} p_n < \infty \), thus by Borel-Cantelli the required prob. = 0