Problem 1.

(a) For \( n = 1 \), \( \varphi_{1,1}(-1) = \frac{1}{2} \); \( E X_{1,1} = 0 \); \( EX_{1,1}^{2} = 1 \); \( \text{Var} \ X_{1,1} = 1 \).

The same answer for \( n \geq 2 \):

\[ \text{Ex}_{n,k} = 0; \text{Ex}_{n,k}^{2} = 1; \text{Var} \ X_{n,k} = 1 \text{ for all } k = 1, 2, \ldots, n. \]

(b) Clearly \( ES_{n} = 0 \); \( \text{Var} \ S_{n} = n \), so \( E \frac{S_{n}}{n} = 0 \). From WLLN \( \frac{S_{n}}{n} \overset{P}{\rightarrow} 0 \).

(c)

\[ E e^{it \frac{S_{n}}{\sqrt{\text{Var} \ S_{n}}}} = \prod_{k=1}^{n} e^{i t \frac{X_{n,k}}{\sqrt{n}}} = \left( \frac{e^{it} + e^{-it}}{2} \right)^{n} \]

\[ = \left( \cos \frac{t}{n} + 1 - \frac{1}{n} \right)^{n} = \left( 1 - \frac{1 - \cos t}{n} \right)^{n} \]

\[ \lim_{n \to \infty} E e^{it \frac{S_{n}}{\sqrt{\text{Var} \ S_{n}}}} = e^{-\frac{1}{2} \cos t}, \quad t \in \mathbb{R} \]

Next, recall that the characteristic function of \( \text{Poi}(\lambda) \) is

\[ \varphi_{\lambda}(t) = e^{-\lambda(1 - e^{it})}, \quad t \in \mathbb{R} \]

and thus

\[ e^{-2\lambda(1 - \cos t)} = \varphi_{\lambda}(t) \varphi_{\lambda}(-t) = 1 \]

This gives

\[ e^{-\frac{1}{2} \cos t} = \varphi_{\frac{1}{2}}(t) \varphi_{\frac{1}{2}}(-t) \]

For any 2 RVs \( \xi, \eta \) we have \( \varphi_{\xi + \eta}(s) = \varphi_{\xi}(s) \varphi_{\eta}(-s) \).

Thus, by (1), (2) \( R_{n} \overset{d}{\to} X_{\frac{1}{2}}' - X_{\frac{1}{2}}'' \), where \( X_{\frac{1}{2}}', X_{\frac{1}{2}}'' \) are independent \( \text{Poi}(\frac{1}{2}) \) random variables.
Problem 2

\[ E X_t = E \left[ \xi(t)^{N(t)} \right] = E \xi E(t)^N = 0. \]

Suppose that \( s < t \), then

\[ E[X_s X_t] = E \sum_{k=0}^{\infty} (-1)^k \lambda(t-s) \frac{(N(s)-N(t))^k}{k!} e^{-\lambda(t-s)} \]

Likewise if \( t < s \), \( E X_s X_t = e^{-2\lambda(s-t)} \), so altogether

\[ E X_s X_t = e^{-2\lambda|s-t|}. \]

Problem 3

The sequence \( (Z_k)_k \) forms a Markov chain. Indeed, by definition for any realization \( Z_1 = 1, Z_2 = n_2, \ldots, Z_k = n_k \) we have:

\[ X_1 \leq X_{n_2} \leq X_{n_3} \leq \ldots \leq X_{n_k}. \]

Next

\[ P(Z_k = n_{k-1} + m \mid Z_{k-1} = n_{k-1}) = P(Z_{k-1} = n_{k-1}, X_{n_{k-1} + m} > X_{n_{k-1}}, X_{n_{k-1} + m - 1} < X_{n_{k-1}}, \ldots, X_{n_{k-1} + 1} < X_{n_{k-1}}) \]

\[ = P(Z_{k-1} = n_{k-1}) \]

Further, the event \( Z_k = n_{k-1} + m \) conditional on \( Z_{k-1} = n_{k-1} \)

is given by

\[ X_{n_{k-1} + m} \geq X_{n_{k-1}}, X_{n_{k-1} + m - 1} < X_{n_{k-1}}, \ldots, X_{n_{k-1} + 1} < X_{n_{k-1}}. \]

For any value \( X_{n_{k-1}} = x \), \( P(X_{n_{k-1} + m} < x) = (F(x))^{m-1} \) by independence,
and \( P(X_{n_{k-1}+m} > x) = 1 - F(x) \).

Then the transition probability

\[
P(Z_k = n_{k-1} + m \mid Z_{k-1} = n_{k-1}) = \int_{-\infty}^{\infty} dF(\xi) (F(\xi) - F(x))^{m-1}
\]

\[
= \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}, \quad m = 1, 2, \ldots
\]

This PMF does not depend on \( k \). Further, clearly

\[ P(Z_k < k) = 0, \]

the first \( k \) elements in the \( k \)th row of the matrix of transitions = 0.

We conclude that the matrix has the form

\[
\begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \ldots & \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \ldots \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{bmatrix}
\]

### Problem 4

We assume, as many students correctly did, that \( X \) and \( Y \) are jointly Gaussian.

(a) Since \( \max(a, b) = \frac{1}{2}(a + b + |a - b|) \) for any two numbers \( a, b \),

\[
E \max(X, Y) = \frac{1}{2} E \left| X - Y \right|
\]

\( Z = X - Y \) is a Gaussian RV with mean 0 and variance \( E(X-Y)^2 = 2 - 2 \rho \).

Thus \( E \max(X, Y) = \frac{1}{2} E |Z| \)

For an \( \mathcal{N}(0, \sigma^2) \) RV \( U 

\[
E|U| = \frac{2}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \int_{-u/2\sigma^2}^{u/2\sigma^2} u \, du = \frac{1}{\sqrt{2\pi} \sigma} \int_{0}^{\infty} e^{-t^2/2\sigma^4} \, dt = 2\sigma
\]

\[
E^{1/2} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/(2\sigma^4)} \, dt = \frac{2\sigma}{\sqrt{2\pi}}
\]
Finally, \( E \max(X,Y) = \frac{1}{2} E|Z| = \sqrt{\frac{2(1-p)}{2\pi}} = \sqrt{\frac{1-p}{\pi}} \)

(b) Since \( f_{XY}(x,y) = \frac{1}{\sqrt{2\pi(1-p^2)}} \exp(-\frac{x^2 - 2pxy + y^2}{2(1-p^2)}) \), we obtain

\[
 f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi(1-p^2)}} e^{-\frac{x^2 - 2pxy + y^2}{2(1-p^2)}}
\]

This is a Gaussian pdf with mean \( pY \).

Since this is true for any realization \( Y=y \), we finally obtain

\[
 E[X|Y] = pY; \quad \text{Var}(X|Y) = 1-p^2.
\]

Another solution of (b):

Notice that \( X-pY \) and \( Y \) are uncorrelated:

\[
 E[(X-pY)Y] = E(XY) - pEY^2 = p - p \cdot 1 = 0
\]

Since they are Gaussian, they are also independent.

Then

\[
\]

For the variance \( \text{Var}(X|Y) \) compute

\[
 \text{Var}(X|Y) = E[X^2|Y] - (E[X|Y])^2
\]

\[
\]

Thus

\[
\]

\[
 \therefore \quad E[X^2|Y] = 1-p^2 + p^2 Y^2
\]

\[
 \text{Var}(X|Y) = 1-p^2 + p^2 Y^2 - p^2 Y^2 = 1-p^2
\]
(c) It is possible to solve this question by computing \( f_{Xz}(x,z) \) where \( Z = X + Y \) is a Gaussian \( N(0, 2 + 2\rho) \) random variable. We can also use the approach in (b), finding a pair of jointly Gaussian RVs that yield the answer. The RVs \( X, Z \) are jointly Gaussian:

\[
E[X] = 0; \quad E[Z] = 0; \quad E[X^2] = 1; \quad E[Z^2] = E[X^2 + 2XY + Y^2] = 2 + 2\rho
\]

\[
E[XZ] = 1 + \rho.
\]

Let \( Z' = \frac{Z}{\sqrt{2(1+\rho)}} \), then \( E[Z'] = 1 \) and \( E[XZ'] = \sqrt{\frac{1+\rho}{2}} \).

Now part (b) applies to RVs \( X \) and \( Z' \), implying:

\[
E(X \mid Z') = \frac{1+\rho}{2} Z'
\]

Let \( Z' = \frac{Z}{\sqrt{2(1+\rho)}} \) \( \iff \) \( X + Y = Z \)

We obtain:

\[
E(X \mid X + Y = z) = \frac{1+\rho}{2} \frac{z}{\sqrt{2(1+\rho)}} = \frac{z}{2}
\]

Similarly, we can use the result of part (b) to compute the variance. The role of \( (X, Y) \) in part (b) is played by \( (X, Z') \), and we obtain:

\[
\text{Var}(X \mid X + Y = z) = 1 - \left( \frac{1+\rho}{2} \right)^2 = \frac{1-\rho}{2}
\]

(d) **First solution:** Direct calculation using

\[
E[X + Y \mid X > 0, Y > 0] = \frac{1}{P(X > 0, Y > 0)} \int_0^\infty \int_0^\infty (x+y) f_{XY}(x,y) \, dx \, dy
\]

where the joint PDF \( f_{XY} \) is given in part (b).

For the ease of calculation we need to separate the variables \( x, y \).
In [Lee.22] we showed that any Gaussian vector can be transformed to a vector of uncorrelated Gaussian RVs by a unitary transformation. Let
\[ \Lambda = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]
then
\[ \Lambda \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Lambda^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+\rho & 1-\rho \\ 1+\rho & -1+\rho \end{bmatrix} = \begin{bmatrix} 2(1+\rho) & 0 \\ 0 & 2(1-\rho) \end{bmatrix} \]

Denote the new variables by \( Z_1, Z_2 \):
\[ Z_1 = X + \gamma, \quad Z_2 = X - \gamma \]

We have
\[ \{X > 0, \gamma > 0\} \iff \{Z_1 + Z_2 > 0, Z_1 - Z_2 > 0\} \]
\[ \iff \{Z_1 > 0, Z_1 > |\gamma| \leq Z_1\} \]

Then
\[ P(X > 0, \gamma > 0) = P(Z_1 > 0, |\gamma| < Z_1) = \int_0^\infty \int_{-Z_1}^{Z_1} \frac{1}{2\pi \sqrt{4-\gamma^2}} e^{-\frac{z_1^2}{4(1+\rho)} - \frac{z_2^2}{4(1-\rho)}} dz_1 dz_2 \]
\[ = \frac{1}{2\pi \sqrt{4-\gamma^2}} \int_0^\infty \int_{-Z_1}^{Z_1} e^{-\frac{z_1^2}{4(1+\rho)} - \frac{z_2^2}{4(1-\rho)}} dZ_2 dZ_1 \]
\[ = \frac{1}{2\pi \sqrt{4-\gamma^2}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^\infty e^{-\frac{r^2 \cos^2 \theta}{4(1+\rho)} - \frac{r^2 \sin^2 \theta}{4(1-\rho)}} r dr d\theta \]
Now using A above, we compute
\[
E(X+Y | X > 0, Y > 0) = E(Z, | Z_1 > 0, \ |Z_2| < Z_1)
\]
\[
= \frac{4}{1+p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z_1^2}{\sqrt{1-p^2}} \frac{z_2^2}{\sqrt{1-(1-p)^2}} \ e^{-\frac{z_1^2}{1+p}} \ e^{-\frac{z_2^2}{1-(1-p)^2}} \ dz_2 \ dz_1 = 2\sqrt{\frac{2}{\pi}}.
\]

Second solution. Let \( A = \{X > 0, Y > 0\} \)
\[
P(A) = \frac{1}{2}, \quad \text{(by symmetry or by direct calculation)}
\]
\[
f_{x1\mid A}(x) = \frac{1}{P(A)} f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad 0 \leq x < \infty
\]
\[
E[X \mid A] = \int_0^\infty x f_{x1\mid A}(x) \ dx = \sqrt{\frac{2}{\pi}} = E[Y \mid A] \quad \text{by symmetry}
\]
Then
\[
E[X + Y \mid A] = 2\sqrt{\frac{2}{\pi}}
\]
by linearity of expectation.

Problem 5

(a) We define a natural filtration \( \cF_n := \sigma(Z_1, \ldots, Z_n) \)
\[
E[X_n | \cF_n] = -\frac{3}{4} X_{n-1} + \frac{c}{4} X_{n-1} = X_{n-1} \quad \text{if} \ c = 3.
\]

(b) The random walk that makes 3 steps to the right a prob. \( \frac{3}{4} \)
and one step to the left with prob. \( \frac{1}{4} \) does not converge a.s.
First note that \( EZ = 0 \); \( EZ^2 = \frac{3}{4} + \frac{9}{4} = 3 \).

Then \( EX_n = 5 \), \( \text{Var} \ X_n = 3n \).

Using CLT, \[
\frac{X_n - 5}{\sqrt{3n}} \xrightarrow{d} N(0,1)
\]

Now suppose that there is an RV \( Y \) s.t. \( X_n \xrightarrow{a.s.} Y \). If so, then \( \frac{X_n}{\sqrt{3n}} \xrightarrow{a.s.} 0 \), which would imply that \( \frac{X_n}{\sqrt{3n}} \xrightarrow{d} 0 \).

This yields a contradiction.

(c) The random walk described in the problem forms an irreducible Markov chain on \( \mathbb{Z} \) that starts at \( X_0 = 5 \). The chain can return to 5 after \( 4n \) steps, \( n \in \mathbb{N} \). Since to compensate one step \( \rightarrow \) we need 3 steps \( \leftarrow \). Thus

\[
P_{5,5}^{(4n)} = \binom{4n}{n} \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^{3n}
\]

\[
\sum_{n=1}^{\infty} \binom{4n}{n} \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^{3n} = \infty \quad \begin{cases} \text{Just barely;} \\ \text{and the inequality is close.} \end{cases}
\]

and thus the states are recurrent.

We showed in class that if state 0 is recurrent (it is) and it can reach 5 with prob. \( f_{0 \rightarrow 5} > 0 \) (it can), then \( f_{5 \rightarrow 0} = \mathbb{P} \left( \bigcap_{n=1}^{\infty} \{ X_n = 0 \} \mid X_0 = 5 \right) = 1 \) (Gallager, Lemma 6.2.4).

Thus \( f_{5,0} = 1 \), so the claim in the question is true.
Another solution:

Set up a gambler's ruin problem with starting capital 0 and the barriers \( a = 5 \) and some \( b > 5 \).

Our random walk forms a martingale, and

\[
\tau := \min \{ n : X_n = -a \text{ or } X_n = b \}
\]

is a finite stopping time, so the Optional Stopping Theorem applies.

Using the example in Lec. 26 (p. 7), the probability of hitting \(-a\) before \(b\) is

\[
\frac{b}{a+b}
\]

Now let \( b \to \infty \) to conclude that the probability of reaching \(-a\) becomes \( = 1 \).