Problem 1.

(a) For $n=1$, $\therefore\ p_{1,1}(-1)=p_{1,1}(1)=\frac{1}{2}$; $EX_{1,1}=0$; $EX_{1,1}^2=1$; $Var\,X_{1,1}=1$

The same answer for $n\geq 2$:

$$EX_{n,k}=0;\ EX_{n,k}^2=1;\ Var\,X_{n,k}=1 \text{ for all } k=1,2,\dots,n$$

(b) Clearly $ES_n=0$; $Var\,S_n=n$, so $E\frac{S_n}{n}=0$. From WLLN $\frac{S_n}{n}\xrightarrow{P}0$.

(c)

$$Ee^{itR_n}=Ee^{it\frac{S_n}{\sqrt{Var S_n}}}=\prod_{k=1}^{n}e^{it\frac{X_{n,k}}{\sqrt{n}}}=\left(\frac{e^{it}+e^{-it}}{2n}+1-\frac{1}{n}\right)^n$$

$$=\left(\frac{\cos t}{n}+1-\frac{1}{n}\right)^n=\left(1-\frac{1-\cos}{n}\right)^n$$

$$\lim_{n\to\infty}Ee^{itR_n}=e^{-(1-\cos t)},\quad t\in\mathbb{R}$$

Next, recall that the characteristic function of $Poi(\lambda)$ is

$$\varphi_\lambda(t)=e^{-\lambda(1-e^{it})},\quad t\in\mathbb{R}$$

and thus

$$e^{-2\lambda(1-\cos t)}=\varphi_\lambda(t)\,\varphi_\lambda(-t)\qquad(1)$$

This gives

$$e^{-(1-\cos t)}=\varphi_{\frac{1}{2}}(t)\,\varphi_{\frac{1}{2}}(-t)$$

For any 2 RVs ξ,η we have $\varphi_{\xi-\eta}(s)=\varphi_\xi(s)\,\varphi_\eta(-s)$. (2)

Thus, by (1), (2) $\boxed{R_n\overset{d}{\Rightarrow}X'_{\frac{1}{2}}-X''_{\frac{1}{2}}}$, where $X'_{\frac{1}{2}}, X''_{\frac{1}{2}}$ are independent $Poi(\frac{1}{2})$ random variables.

Problem 2

$$E X_t = E\left[\xi (-1)^{N(t)}\right] = E\xi \, E(-1)^{N(t)} = 0.$$

Suppose that $s<t$, then

$$E[X_s X_t] = \underbrace{E\xi^2}_{1} E\left[(-1)^{N(s)+N(t)}\right] = E(-1)^{N(s)-N(t)}$$

$$= \sum_{k=0}^{\infty} (-1)^k P(N(s)-N(t)=k) = \sum_{k=0}^{\infty} (-1)^k \frac{\lambda(t-s)^k}{k!} e^{-\lambda(t-s)}$$

$$= e^{-2\lambda(t-s)}.$$

Likewise if $t<s$, $E X_s X_t = e^{-2\lambda(s-t)}$, so altogether

$$E X_s X_t = e^{-2\lambda|s-t|}.$$

Problem 3

The sequence $(Z_k)_k$ forms a Markov chain. Indeed, by definition for any realization $Z_1=1, Z_2=n_2, \dots, Z_k=n_k$ we have.

we have $n_1=1<n_2<\dots<n_k$

$$X_1 \le X_{n_2} \le X_{n_3} \le \dots \le X_{n_k}.$$

Next

$$P(Z_k = n_{k-1}+m \,/\, Z_1^{k-1} = n_1^{k-1})$$

$$= \frac{P(Z_1^{k-1}=n_1^{k-1};\ X_{n_{k-1}+m} \geq X_{n_{k-1}};\ X_{n_{k-1}+m-1} < X_{n_{k-1}}, \dots, X_{n_{k-1}+1} < X_{n_{k-1}})}{P(Z_1^{k-1}=n_1^{k-1})}$$

$$= P(Z_k = n_{k-1}+m \,/\, Z_{k-1}=n_{k-1}).$$

Further, the event $Z_k = n_{k-1}+m$ conditional on $Z_{k-1}=n_{k-1}$ is given by

$$X_{n_{k-1}+m} \geq X_{n_{k-1}};\ X_{n_{k-1}+1}^{n_{k-1}+m-1} < X_{n_{k-1}}$$

For any value $X_{n_{k-1}}=x$, $P(X_{n_{k-1}+1}^{n_{k-1}+m-1} < x) = (F(x))^{m-1}$ by independence, and $P(X_{n_{k-1}+m} \geq x) = 1 - F(x)$.

Then the transition probability out

$$P(Z_k = n_{k-1}+m \mid Z_{k-1} = n_{k-1}) = \int_{-\infty}^{\infty} dF(x)\,(F(x))^{m-1}(1-F(x))$$

$$= \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}, \quad m = 1, 2, \ldots$$

This PMF does not depend on k. Further, clearly

$$P(Z_k < k) = 0,$$

the first k elements in the k^{th} row of the matrix of transitions $= 0$.

We conclude that the matrix has the form

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \cdots & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\cdot 3} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

Problem 4. We assume, as many students correctly did, that X and Y are jointly Gaussian.

(a) Since $\max(a,b) = \frac{1}{2}(a+b+|a-b|)$ for any two numbers a, b,

$$E\max(X,Y) = \frac{1}{2}E|X-Y|$$

$Z := X-Y$ is a Gaussian RV with mean 0 and variance $E(X-Y)^2 = 2-2\rho$

Thus $\quad E\max(X,Y) = \frac{1}{2}E|Z|$

For an $\mathcal{N}(0,\sigma^2)$ RV U

$$E|U| = \frac{2}{\sqrt{2\pi}\,\sigma}\int_0^\infty e^{-u^2/2\sigma^2}u\,du = \frac{1}{\sqrt{2\pi}\,\sigma}\int_0^\infty e^{-\frac{t}{2\sigma^2}}dt = \frac{2\sigma}{\sqrt{2\pi}}$$

Finally, $E\max(X,Y) = \frac{1}{2}E|Z| = \sqrt{\frac{2(1-\rho)}{2\pi}} = \sqrt{\frac{1-\rho}{\pi}}$

(b) Since $f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2(1-\rho^2)}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)$, we obtain

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)} + \frac{y^2}{2}} = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}}$$

This is a Gaussian pdf with mean ρy

Since this is true for any realization $Y=y$, we finally obtain

$$E[X|Y] = \rho Y; \quad Var(X|Y) = 1-\rho^2.$$

Another solution of (b):

Notice that $X-\rho Y$ and Y are uncorrelated:

$$E[(X-\rho Y)Y] = EXY - \rho EY^2 = \rho - \rho\cdot 1 = 0$$

Since they are Gaussian, they are also independent.

Then

$$0 = E[X-\rho Y] = E[(X-\rho Y)|Y] = E[X|Y] - \rho E[Y|Y] =$$
$$= E[X|Y] - \rho Y.$$

For the variance $Var(X|Y)$ compute

$$Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

$$E[(X-\rho Y)^2|Y] = E[(X-\rho Y)^2] = 1-2\rho^2+\rho^2 = 1-\rho^2$$ " $Y \cdot (\rho Y)$

Thus

$$1-\rho^2 = E[(X-\rho Y)^2|Y] = E[X^2|Y] - 2\rho E[XY|Y] + \rho^2 E[Y^2|Y]$$
$$= E[X^2|Y] - 2\rho^2 Y^2 + \rho^2 Y^2 = E[X^2|Y] - \rho^2 Y^2$$

$$\therefore \; E[X^2|Y] = 1-\rho^2+\rho^2 Y^2$$

$$Var(X|Y) = 1-\rho^2+\rho^2Y^2-\rho^2Y^2 = \boxed{1-\rho^2}$$

(c) It is possible to solve this question by computing $f_{XZ}(x,z)$ where $Z = X+Y$ is a Gaussian $\mathcal{N}(0, 2+2\rho)$ random variable.

We can also use the appoach in (b), finding a pair of jointly Gaussian RVs that yield the answer.

The RVs X, Z are jointly Gaussian

$$EX=0;\ EZ=0;\ EX^2=1;\ EZ^2 = E[X^2+2XY+Y^2] = 2+2\rho$$

$$E[XZ] = 1+\rho.$$

Let $Z' = \dfrac{Z}{\sqrt{2(1+\rho)}}$, then $EZ'^2 = 1$ and $E[XZ'] = \sqrt{\dfrac{1+\rho}{2}}$.

Now part (b) applies to RVs X and Z', implying

$$E(X \mid Z') = \sqrt{\frac{1+\rho}{2}}\, Z'$$

Let $z' = \dfrac{z}{\sqrt{2(1+\rho)}} \iff X+Y = z$

We obtain $E(X \mid X+Y = z) = \sqrt{\dfrac{1+\rho}{2}}\, \dfrac{z}{\sqrt{2(1+\rho)}} = \boxed{\dfrac{z}{2}}$

Similarly, we can use the result of Part (b) to compute the variance. The role of (X,Y) in part (b) is played by (X, Z'), and we obtain

$$\mathrm{Var}(X \mid X+Y = z) = 1 - \left(\sqrt{\frac{1+\rho}{2}}\right)^2 = \frac{1-\rho}{2}$$

(d) **First solution :** Direct calculation using

$$E[X+Y \mid X>0, Y>0] = \frac{1}{P(X>0, Y>0)} \int_0^\infty\!\!\int_0^\infty (x+y)\, f_{XY}(x,y)\, dx\, dy \qquad \text{A}$$

where the joint PDF f_{XY} is given in Part (b).

For the ease of calculation we need to separate the variables x, y.

In Lec. 22 we showed that any Gaussian vector can be transformed to a vector of uncorrelated Gaussian RVs by a unitary transformation. Let

$$\Lambda = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then

$$\Lambda \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \Lambda^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+\rho & 1-\rho \\ 1+\rho & -1+\rho \end{bmatrix} = \begin{bmatrix} 2(1+\rho) & 0 \\ 0 & 2(1-\rho) \end{bmatrix}$$

Denote the new variables by Z_1, Z_2:

$$Z_1 = X+Y, \quad Z_2 = X-Y$$

We have

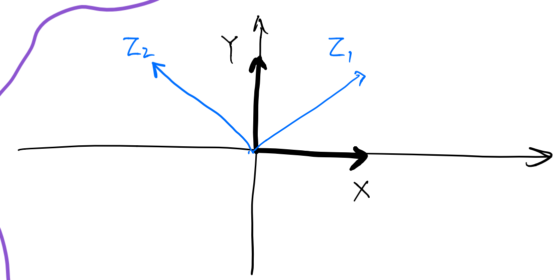
$$\{X>0, Y>0\} \iff \{Z_1+Z_2>0 \,;\, Z_1-Z_2>0\}$$

$$\iff \{Z_1>0 \,;\, |Z_2|<Z_1\}$$

Then

$$P(X>0, Y>0) = P(Z_1>0, |Z_2|<Z_1) = \int_0^\infty \int_{-z_1}^{z_1} f_{Z_1Z_2}(z_1,z_2)\, dz_1\, dz_2$$

$$= \frac{1}{2\pi\sqrt{4-4\rho^2}} \int_0^\infty \int_{-z_1}^{z_1} e^{-\frac{z_1^2}{4(1+\rho)} - \frac{z_2^2}{4(1-\rho)}}\, dz_2\, dz_1$$



Change of variables.

$$= \frac{1}{2\pi\sqrt{4-4\rho^2}} \int_{\pi/4}^{3\pi/4} \int_0^\infty e^{-\frac{r^2\cos^2\theta}{4(1+\rho)} - \frac{r^2\sin^2\theta}{4(1-\rho)}}\, r\, dr\, d\theta$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi} \int_{\pi/4}^{3\pi/4} \frac{d\theta}{1-\rho\cos\theta} = \frac{1+\rho}{4}$$

Now using (A) above, we compute

$$E(X+Y \mid X>0, Y>0) = E(Z_1 \mid Z_1>0, |Z_2|<Z_1)$$

$$= \frac{4}{1+\rho} \int_{-\infty}^{\infty} \int_{|Z_1|}^{\infty} \frac{z_1}{4\pi\sqrt{1-\rho^2}} e^{-\frac{z_1^2}{4(1+\rho)} - \frac{z_2^2}{4(1-\rho)}} dz_2 \, dz_1 = 2\sqrt{2/\pi}.$$

Second solution. Let $A = \{X>0, Y>0\}$

$P(A) = \frac{1}{2}$ (By symmetry or by direct calculation)

$$f_{X|A}(x) = \frac{1}{P(A)} f_X(x) = 2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad 0 \leq x < \infty$$

$$E[X|A] = \int_0^\infty x f_{X|A}(x) dx = \sqrt{\frac{2}{\pi}} = E[Y|A] \text{ by symmetry}$$

Then $E[X+Y|A] = 2\sqrt{\frac{2}{\pi}}$

by linearity of expectation.

Problem 5

(a) We define a natural filtration $\mathcal{F}_n := \sigma(Z_1, \dots, Z_n)$

$$E[X_n \mid \mathcal{F}_n] = -\frac{3}{4} X_{n-1} + \frac{c}{4} X_{n-1} = X_{n-1} \text{ if } c=3.$$

(b) The random walk that makes 3 steps to the right w. prob. 1/4 and one step to the left with prob. 3/4 does not converge a.s.

First note that $EZ = 0$; $EZ^2 = \frac{3}{4} + \frac{9}{4} = 3$.

Then $EX_n = 5$, $\operatorname{Var} X_n = 3n$.

Using CLT,

$$\frac{X_n - 5}{\sqrt{3n}} \xrightarrow{d} N(0,1)$$

Now suppose that there is an RV Y s.t. $X_n \xrightarrow{a.s.} Y$. If so, then $\frac{X_n}{\sqrt{3n}} \xrightarrow{a.s.} 0$, which would imply that $\frac{X_n}{\sqrt{3n}} \xrightarrow{d} 0$.

This yields a contradiction.

(c) The random walk described in the problem forms an irreducible Markov chain on $\mathbb{Z}$ that starts at $X_0 = 5$. The chain can return to 5 after $4n$ steps, $n \in \mathbb{N}$ since to compensate one step $\boxed{\rightarrow}$ we need 3 steps $\boxed{\leftarrow}$. Thus

$$P^{(4n)}_{5,5} = \binom{4n}{n}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^{3n}$$

$$\sum_n P^{(4n)}_{5,5} = \sum_{n=1}^{\infty} \binom{4n}{n}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^{3n} = \infty$$

Just barely; $\binom{4n}{n}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^{3n} > \frac{1}{\sqrt{\frac{3}{2}n}}$ and the inequality is close.

and thus the states are recurrent.

We showed in class that if state 0 is recurrent (it is) and it can reach 5 with prob. $f_{0\to 5} > 0$ (it can), then $f_{5\to 0} := P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_0 = 5\right) = 1$ (Gallager, Lemma 6.2.4)

Thus $f_{5,0} = 1$, so the claim in the question is true.

Another solution:

Set up a gambler's ruin problem with starting capital 0 and the barriers $a = 5$ and some $b > 5$.

Our random walk forms a martingale and

$$\tau := \min\{n : X_n = -a \text{ or } X_n = b\}$$

is a finite stopping time, so the Optional Stopping Theorem applies.

Using the example in Lec. 26 (p. 7), the probability of hitting $-a$ before b is

$$\frac{b}{a+b}$$

Now let $b \to \infty$ to conclude that the probability of reaching $-a$ becomes $= 1$.