

- Closed book; one two-sided sheet of handwritten notes is permitted. Calculators are permitted.
- Justification is required: a numerical answer with no reasoning receives no credit.
- Exact answers (fractions, powers, binomial coefficients) are acceptable everywhere; you need not evaluate them as decimals.
- **Five problems**, each problem is worth 20 points, for a total of 100.

Problem 1. Five fair six-sided dice are rolled; all 6^5 ordered outcomes are equally likely.

- What is the probability that all five dice show different faces?
- What is the probability that at least two of the dice show the same face?
- What is the probability that the five dice show *exactly two* distinct faces (for example 3, 3, 5, 3, 5)?

Solution. There are $6^5 = 7776$ equally likely outcomes.

- (a) Fill the five dice left to right with distinct faces: $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$ outcomes, so

$$P(\text{all different}) = \frac{720}{7776} = \frac{5}{54} \approx 0.0926.$$

- (b) This is the complement of (a): $P(\text{at least two equal}) = 1 - \frac{5}{54} = \frac{49}{54} \approx 0.9074$.

(c) Choose the two faces that appear, $\binom{6}{2} = 15$ ways. Each of the five dice must show one of these two faces, and both faces must actually occur, so the number of assignments is $2^5 - 2 = 30$ (all 2^5 patterns, minus the two constant ones). Hence

$$P(\text{exactly two distinct faces}) = \frac{15 \cdot 30}{7776} = \frac{450}{7776} = \frac{25}{432} \approx 0.0579.$$

Problem 2. A drawer holds three coins: two are fair, and the third is two-headed (both sides H). You pick a coin at random without learning its type, and toss it twice. Both tosses come up heads.

- What is the probability of observing two heads?
- Given the two heads, what is the probability that you picked the two-headed coin?
- Given the two heads, what is the probability that a third toss of the *same* coin is also a head?
- Suppose instead the two tosses had come out H and then T. What is the probability that you picked the two-headed coin? Answer in one line, without computation.

Solution. Let F be the event that the chosen coin is fair and D that it is the two-headed one, so $P(F) = \frac{2}{3}$ and $P(D) = \frac{1}{3}$. Let HH be the event of two heads; the likelihoods are $P(HH | F) = \frac{1}{4}$ and $P(HH | D) = 1$.

- (a) By the law of total probability,

$$P(HH) = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 1 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

- (b) By Bayes' rule,

$$P(D | HH) = \frac{P(HH | D)P(D)}{P(HH)} = \frac{1/3}{1/2} = \frac{2}{3}, \quad \text{so} \quad P(F | HH) = \frac{1}{3}.$$

Seeing two heads doubles the odds on the two-headed coin, from 1:2 to 2:1.

- (c) Condition on which coin is in hand, using the posterior from (b):

$$P(3\text{rd is H} | HH) = P(F | HH) \cdot \frac{1}{2} + P(D | HH) \cdot 1 = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}.$$

(d) Zero: the two-headed coin cannot produce a tail, so the observation rules it out ($P(HT \mid D) = 0$ makes the numerator in Bayes' rule vanish).

Problem 3. A fair four-sided die is rolled twice; let X and Y be the two outcomes, so all 16 pairs (X, Y) with $X, Y \in \{1, 2, 3, 4\}$ are equally likely. Define

$$A = \{X \text{ is odd}\}, \quad B = \{Y \text{ is odd}\}, \quad C = \{X + Y \text{ is odd}\}.$$

(a) Compute $P(A)$, $P(B)$, $P(C)$ and the three pairwise intersection probabilities, and decide whether the events are pairwise independent.

(b) Are A , B , C jointly independent? Justify.

(c) Are A and C conditionally independent given B ? Justify.

Solution. (a) Each of X , Y is odd with probability $\frac{1}{2}$, so $P(A) = P(B) = \frac{1}{2}$. The sum is odd exactly when the two rolls have opposite parity, which happens in 8 of the 16 pairs, so $P(C) = \frac{1}{2}$. For the pairs,

$$P(AB) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad (\text{different rolls}),$$

$$P(AC) = P(X \text{ odd}, Y \text{ even}) = \frac{1}{4}, \quad P(BC) = P(Y \text{ odd}, X \text{ even}) = \frac{1}{4}.$$

Each equals the product of the two individual probabilities, so the three events are pairwise independent.

(b) No. If X and Y are both odd then $X + Y$ is even, so $A \cap B \cap C = \emptyset$ and

$$P(ABC) = 0 \neq \frac{1}{8} = P(A)P(B)P(C).$$

Pairwise independence does not imply joint independence: the three-fold product is a separate requirement.

(c) No. Given B , i.e. given that Y is odd, the event C becomes $\{X \text{ is even}\}$, which is exactly the complement of A . Hence

$$P(AC \mid B) = 0, \quad \text{while} \quad P(A \mid B)P(C \mid B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Conditioning on B destroys the independence of A and C .

Problem 4. A box contains five chips numbered 1, 2, 3, 4, 5. Two chips are drawn at random without replacement, all $\binom{5}{2}$ pairs being equally likely. Let M be the larger of the two numbers drawn and let D be the (positive) difference of the two numbers.

(a) Find the range of M and the pmf $p_M(m)$, and verify that it sums to 1.

(b) Write the CDF $F_M(x)$ for all real x , and use it to compute $P(2 < M \leq 4)$.

(c) Are M and D independent? Justify with a single computation.

Solution. There are $\binom{5}{2} = 10$ equally likely pairs.

(a) The event $\{M = m\}$ consists of the pairs whose larger element is m ; the smaller one may be any of $1, \dots, m-1$. Hence $\text{Range}(M) = \{2, 3, 4, 5\}$ and

$$p_M(m) = \frac{m-1}{10}, \quad \text{that is} \quad \begin{array}{c|cccc} m & 2 & 3 & 4 & 5 \\ \hline p_M(m) & \frac{1}{10} & \frac{2}{10} & \frac{3}{10} & \frac{4}{10} \end{array}$$

and $\frac{1+2+3+4}{10} = 1$. Equivalently $P(M \leq m) = \binom{m}{2} / \binom{5}{2}$, and

$$p_M(m) = P(M \leq m) - P(M \leq m-1) = \binom{m}{2} - \binom{m-1}{2} = m-1.$$

(b) Accumulating the masses,

$$F_M(x) = \begin{cases} 0, & x < 2, \\ 1/10, & 2 \leq x < 3, \\ 3/10, & 3 \leq x < 4, \\ 6/10, & 4 \leq x < 5, \\ 1, & x \geq 5. \end{cases}$$

Therefore $P(2 < M \leq 4) = F_M(4) - F_M(2) = \frac{6}{10} - \frac{1}{10} = \frac{1}{2}$, which agrees with $p_M(3) + p_M(4) = \frac{2}{10} + \frac{3}{10}$.

(c) No. Take $m = 2$, $d = 1$. The only pair with $M = 2$ is $\{1, 2\}$, and it has $D = 1$, so $P(M = 2, D = 1) = \frac{1}{10}$. On the other hand $P(M = 2) = \frac{1}{10}$ and $P(D = 1) = \frac{4}{10}$ (the pairs $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$), so

$$P(M = 2)P(D = 1) = \frac{1}{10} \cdot \frac{4}{10} = \frac{1}{25} \neq \frac{1}{10} = P(M = 2, D = 1).$$

Problem 5. A machine produces items one after another; each item is defective with probability $\frac{1}{4}$, independently of all the others.

(a) Let X be the number of defective items among the 8 items produced. Name the distribution of X and compute $P(X = 2)$.

(b) What is the smallest number n of items that must be produced so that the probability of seeing at least one defective is at least 0.9? (You may use $(3/4)^8 = 0.1001$ and $(3/4)^9 = 0.0751$.)

(c) Let W be the number of good items produced before the first defective one. Name the distribution of W and compute $P(W \geq 3)$.

(d) A finished box contains 10 items, of which 4 are defective. Three items are drawn from the box without replacement; let H be the number of defective items drawn. Name the distribution of H , compute $P(H = 1)$, and explain in one sentence why H is *not* binomial.

Solution. (a) The 8 items are independent trials with the same defect probability, so $X \sim \text{Bin}(8, \frac{1}{4})$ and

$$P(X = 2) = \binom{8}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^6 = 28 \cdot \frac{729}{4^8} = \frac{5103}{16384} \approx 0.311.$$

(b) With n items, $P(\text{at least one defective}) = 1 - (3/4)^n$, which increases in n . From the given values,

$$n = 8: \quad 1 - 0.1001 = 0.8999 < 0.9, \quad n = 9: \quad 1 - 0.0751 = 0.9249 \geq 0.9,$$

so the answer is $n = 9$. (Note how narrowly $n = 8$ fails.)

(c) Counting failures before the first success, $W \sim \text{Geom}(\frac{1}{4})$ with $p_W(k) = (\frac{3}{4})^k \frac{1}{4}$ for $k = 0, 1, 2, \dots$. The event $\{W \geq 3\}$ says the first three items are all good, so

$$P(W \geq 3) = \left(\frac{3}{4}\right)^3 = \frac{27}{64} \approx 0.422.$$

(d) Sampling 3 items without replacement from a box with 4 defective and 6 good items gives a hypergeometric variable, $H \sim \text{HGeom}(4, 6, 3)$, with

$$P(H = 1) = \frac{\binom{4}{1} \binom{6}{2}}{\binom{10}{3}} = \frac{4 \cdot 15}{120} = \frac{1}{2}.$$

It is not binomial because the draws are made without replacement: the composition of the box changes after each draw, so the three draws are not independent and do not share a common defect probability.