ENEE324, Home assignment 6. Date due Saturday Nov. 7, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Notation: $\mathrm{Exp}(\lambda)$ has density $\lambda e^{-\lambda x}$, $x \geq 0$; $\mathcal{N}(\mu, \sigma^2)$ has mean μ and variance σ^2; Φ is the standard $\mathcal{N}(0,1)$ CDF.

Problem 1. The size X (in Mbytes) of a file stored on a server is modeled by the density

$$f(x) = \frac{c}{x^4}, \quad x \geq 1, \qquad f(x) = 0, \quad x < 1.$$

(1) Find c and the CDF $F(x)$.
(2) Compute $\mathbb{E}[X]$ twice: from the definition $\int x f(x)\, dx$, and from the tail formula $\mathbb{E}[X] = \int_0^\infty P(X > x)\, dx$.
(3) Compute $\mathbb{E}[X^2]$ by LOTUS, and $\mathrm{Var}(X)$.
(4) Show that $\mathbb{E}[X^3] = \infty$.
(5) Find the median of X and $P(X > \mathbb{E}[X])$.

Problem 2. Let $U \sim \mathrm{Unif}(0,1)$ and $Y = e^U$.

(1) Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$ using LOTUS.
(2) Compare $\mathbb{E}[Y]$ with $e^{\mathbb{E}[U]}$. Which is larger, and why could you predict this without computing?
(3) Find the CDF and the pdf of Y (state the range of Y).
(4) Compute $\mathbb{E}[Y]$ again, now from the pdf found in part (3).
(5) Find $P\big(Y > \mathbb{E}[Y]\big)$.

Problem 3. (1) Let $Z \sim \mathcal{N}(0,1)$. Show that $\mathbb{E}|Z| = \sqrt{2/\pi}$. (Hint: use the symmetry of the density and the substitution $u = z^2/2$.)
(2) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Using $X = \mu + \sigma Z$, find $\mathbb{E}|X - \mu|$ and $\mathbb{E}[X^2]$.
(3) Compute $P(|X - \mu| \geq 2\sigma)$ and compare it with the upper bound given by Chebyshev's inequality.
(4) The time to drive to campus is $X \sim \mathcal{N}(30, 25)$ minutes. How many minutes before class should you leave so that you arrive on time with probability 0.95?

Problem 4. A bit is sent over a noisy channel as a voltage $+A$ (for 1) or $-A$ (for 0). The receiver observes $Y = \pm A + N$ with noise $N \sim \mathcal{N}(0, \sigma^2)$ and decides by the sign of Y. Take the Signal-to-noise ratio to be $A/\sigma = 2$.

(1) Show that the probability of a wrong decision is $p = \Phi(-A/\sigma)$, and evaluate it.
(2) To improve reliability, the bit is sent 3 times with independent noise, and the receiver decides by majority vote of the three decisions. Find the probability of a wrong decision.
(3) Sending the bit 3 times uses 3 times the energy, and the energy of a pulse is proportional to A^2. Suppose instead the bit is sent once with amplitude $A' = \sqrt{3}\, A$ (the same total energy). Find the probability of a wrong decision and compare with part (2).

Problem 5. A help desk has 3 agents. When you call, all three are busy with other callers. Service times are independent $\mathrm{Exp}(\lambda)$ with mean 4 minutes, and you are next in line.

(1) Find the distribution of your waiting time W until an agent becomes free, its mean, and $P(W > 2)$.
(2) Find the probability that you are the last of the four callers to finish.
(3) Find the expected total time from your call until your own service is complete.
(4) Let T be a lifetime with density $f(t) = 2t\,e^{-t^2}$, $t \geq 0$. Find its survival function $1 - F(t)$ and its hazard rate $h(t) = f(t)/(1 - F(t))$. Is T memoryless? Compute $P(T > s + 1 \mid T > s)$ to support your answer.

Problem 6. E-mails arrive in your inbox as a Poisson process with rate 6 per hour. Let N_t be the number of e-mails in the first t hours and T_k the arrival time of the k-th e-mail.

(1) Find the probability that no e-mail arrives in a given 15-minute interval.
(2) Find the probability that exactly 2 e-mails arrive in the first 10 minutes and exactly 3 in the following 20 minutes.
(3) Find $\mathbb{E}[T_3]$ and $\mathrm{Var}(T_3)$.
(4) Find $P(T_2 \leq 10 \text{ min})$ in two ways: through the event $\{T_2 \leq t\} = \{N_t \geq 2\}$, and by integrating the $\mathrm{Gamma}(2, \lambda)$ density $\lambda^2 x e^{-\lambda x}$.
(5) Given that exactly 4 e-mails arrived during the first hour, find the probability that exactly 1 of them arrived during the first 15 minutes.

Problem 7. (Building a Poisson process from uniforms)

(1) Using universality of the uniform, show that if $U \sim \mathrm{Unif}(0, 1)$ then $T = -\frac{1}{\lambda}\ln(1 - U) \sim \mathrm{Exp}(\lambda)$.
(2) Write a program that generates independent inter-arrival times $T_1', T_2', \ldots$ in this way, with $\lambda = 3$, and records N_1, the number of arrivals in $[0, 1]$ (i.e., the number of partial sums $T_1' + \cdots + T_k'$ that are ≤ 1). Repeat 10^5 times.
(3) Compare the empirical frequencies of $N_1 = 0, 1, \ldots, 8$ with the $\mathrm{Pois}(3)$ pmf (a table or a plot), and compare the sample mean and sample variance of N_1 with their theoretical values. Include your code.