ENEE324, Home assignment 5. Date due November 1, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. The lifetime T (in years) of a solid-state drive is modeled as an exponential random variable, $T \sim \mathrm{Expo}(\lambda)$, with $\mathbb{E}[T] = 4$ years.

(1) Find λ, the pdf and the CDF of T, and $\mathrm{Var}(T)$. Compute $P(T > \mathbb{E}[T])$.
(2) Find the median of T, i.e., the number m with $P(T \le m) = \frac{1}{2}$, and compare it with $\mathbb{E}[T]$.
(3) A drive has been working for 3 years. Find the probability that it works for at least 2 more years, and compare it with the probability that a new drive works for at least 2 years.
(4) A server stores its data on two such drives (a mirrored pair) with independent lifetimes $T_1, T_2 \sim \mathrm{Expo}(\lambda)$. The data are lost only when both drives have failed, i.e., at time $L = \max(T_1, T_2)$. Find the CDF and the pdf of L, and compute $\mathbb{E}[L]$ and $P(L > 4)$.
(5) Is L exponentially distributed? Justify your answer.

Solution. *(1)* $\mathbb{E}[T] = 1/\lambda = 4$, *so* $\lambda = \frac{1}{4}$ *per year. For* $t > 0$,

$$f_T(t) = \tfrac{1}{4}\, e^{-t/4}, \qquad F_T(t) = 1 - e^{-t/4},$$

and $f_T(t) = F_T(t) = 0$ *for* $t \le 0$. *Further,* $\mathrm{Var}(T) = 1/\lambda^2 = 16$. *Finally,* $P(T > 4) = e^{-1} \approx 0.368$, *so about* 63% *of the drives fail before the mean lifetime.*

(2) $1 - e^{-m/4} = \frac{1}{2}$ *gives* $m = 4 \ln 2 \approx 2.77$ *years. The median is smaller than the mean:* $m/\mathbb{E}[T] = \ln 2 \approx 0.69$.

(3) By memorylessness,

$$P(T > 5 \mid T > 3) = \frac{P(T > 5)}{P(T > 3)} = \frac{e^{-5/4}}{e^{-3/4}} = e^{-1/2} \approx 0.607 = P(T > 2).$$

In this model a drive that is still working after 3 *years is as good as a new one. (Real drives also wear out; the exponential model ignores this.)*

(4) For $t > 0$, *by independence,*

$$F_L(t) = P(T_1 \le t,\ T_2 \le t) = \big(1 - e^{-t/4}\big)^2,$$
$$f_L(t) = F_L'(t) = 2\big(1 - e^{-t/4}\big) \cdot \tfrac{1}{4} e^{-t/4} = \tfrac{1}{2}\big(e^{-t/4} - e^{-t/2}\big),$$

and $F_L(t) = f_L(t) = 0$ *for* $t \le 0$. *Since* $1 - F_L(t) = 2e^{-t/4} - e^{-t/2}$, *the tail formula gives*

$$\mathbb{E}[L] = \int_0^\infty \big(2e^{-t/4} - e^{-t/2}\big)\, dt = 8 - 2 = 6 \text{ years}, \qquad P(L > 4) = 2e^{-1} - e^{-2} \approx 0.600.$$

The mirrored pair extends the mean time to data loss from 4 *to* 6 *years.*

(5) No. If $L \sim \mathrm{Expo}(\mu)$ for some $\mu > 0$, then $P(L > t) = e^{-\mu t}$, whose derivative at $t = 0$ is $-\mu \neq 0$. But $\frac{d}{dt}\big(2e^{-t/4} - e^{-t/2}\big)\big|_{t=0} = -\frac{1}{2} + \frac{1}{2} = 0$. Equivalently, $f_L(t) \to 0$ as $t \to 0^+$, whereas an exponential density tends to $\mu > 0$: a new pair almost never loses its data immediately, since both drives must fail.

Problem 2. Let $X \sim \mathrm{FS}(p)$ (First Success), $P(X = k) = p(1-p)^{k-1},\ k = 1, 2, \ldots$, and let

$$W = \mathbf{1}_{\{X \text{ is even}\}}.$$

(1) Explain why $\mathrm{Var}(W) = P(X \text{ even})\big(1 - P(X \text{ even})\big)$.
(2) Compute $P(X \text{ even})$ in closed form.
(3) Deduce a closed-form expression for $\mathrm{Var}(W)$.
(4) Evaluate $\mathrm{Var}(W)$ in the limits $p \to 0$ and $p \to 1$, and briefly interpret.

Solution. *(1) W is a Bernoulli (indicator) random variable with success probability $r = P(X \text{ even})$. Any Bernoulli variable has $\mathbb{E}[W] = r$ and $\mathbb{E}[W^2] = r$, hence $\mathrm{Var}(W) = r - r^2 = r(1-r)$.*

(2) $$P(X \text{ even}) = \sum_{j \geq 1} p(1-p)^{2j-1} = p(1-p)\sum_{j \geq 1}(1-p)^{2(j-1)} = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

(3) With $r = \dfrac{1-p}{2-p}$ we get $1 - r = \dfrac{1}{2-p}$, so

$$\mathrm{Var}(W) = r(1-r) = \frac{1-p}{(2-p)^2}.$$

(4) As $p \to 0$: $\mathrm{Var}(W) \to \frac{1}{4}$. The distribution spreads far out, even and odd values become almost equally likely, so W behaves like a fair coin (maximal variance $1/4$). As $p \to 1$: $\mathrm{Var}(W) \to 0$, since $X = 1$ (odd) with probability approaching 1, making $W \equiv 0$ essentially deterministic.

Problem 3. The resistance R (in ohms) of a resistor taken from a production line is modeled as $R \sim \mathcal{N}(100, 4)$, i.e., with mean $\mu = 100$ and standard deviation $\sigma = 2$. Express your answers through the standard normal CDF Φ, and then evaluate them numerically by computer.

(1) Find $P(98 < R < 103)$.
(2) A resistor is rejected if $|R - 100| > 5$. What fraction of the production is rejected?
(3) Find the value r such that 99% of the resistors have $R \leq r$.
(4) How small must σ be (with $\mu = 100$ unchanged) for the rejection rate in part (2) to be at most 0.1%?
(5) A voltage of 10 V is applied to the resistor, so the dissipated power is $W = 100/R$ watts. Find $P(W > 1.02)$. (The model gives $P(R \leq 0) = \Phi(-50) < 10^{-544}$; you may treat this probability as 0.)

Solution. *Throughout, $Z = (R - 100)/2 \sim \mathcal{N}(0,1)$, so $P(R \leq x) = \Phi\big(\frac{x-100}{2}\big)$, and $\Phi(-z) = 1 - \Phi(z)$.*

(1) $P(98 < R < 103) = \Phi(1.5) - \Phi(-1) = \Phi(1.5) + \Phi(1) - 1 \approx 0.9332 + 0.8413 - 1 = 0.7745$.

(2) $P(|R-100| > 5) = P(|Z| > 2.5) = 2\big(1 - \Phi(2.5)\big) \approx 2 \cdot 0.00621 = 0.0124$, *i.e., about 1.2% of the production.*

(3) We need $\Phi\big(\frac{r-100}{2}\big) = 0.99$, i.e., $\frac{r-100}{2} = \Phi^{-1}(0.99) \approx 2.326$, so $r \approx 104.65$ ohms.

(4) The rejection rate is $P(|R-100| > 5) = 2\big(1 - \Phi(5/\sigma)\big)$. It is at most 0.001 iff $\Phi(5/\sigma) \geq 0.9995$, i.e., $5/\sigma \geq \Phi^{-1}(0.9995) \approx 3.291$, i.e., $\sigma \leq 5/3.291 \approx 1.52$ ohms.

(5) For $R > 0$ the map $R \mapsto 100/R$ is decreasing, so $\{W > 1.02\} = \{0 < R < 100/1.02\}$, with $100/1.02 \approx 98.04$ (for $R \leq 0$ we have $W \leq 0$). Hence

$$P(W > 1.02) = \Phi\Big(\frac{100/1.02 - 100}{2}\Big) - \Phi(-50) \approx \Phi(-0.980) \approx 0.163.$$

Problem 4. In a large network with n links, each link fails during a given day independently with probability $\mathbb{P}(X_i = 1) = \dfrac{\lambda}{n}$ (and $\mathbb{P}(X_i = 0) = 1 - \dfrac{\lambda}{n}$), where $\lambda > 0$ is fixed. Let $S_n = X_1 + \cdots + X_n$ be the number of failed links.

(1) Compute $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$.
(2) Write the exact distribution $\mathbb{P}(S_n = k)$.
(3) Show that for each fixed k, $\mathbb{P}(S_n = k) \longrightarrow \dfrac{e^{-\lambda}\lambda^k}{k!}$ as $n \to \infty$.
(4) Conclude that for large n, S_n is approximately $\mathrm{Poisson}(\lambda)$. Taking $\lambda = 2$, estimate $\mathbb{P}(S_n = 0)$ and $\mathbb{P}(S_n \geq 1)$.

Solution. *(1)* $\mathbb{E}[S_n] = n \cdot \frac{\lambda}{n} = \lambda$*; by independence* $\mathrm{Var}(S_n) = n \cdot \frac{\lambda}{n}\big(1 - \frac{\lambda}{n}\big) = \lambda\big(1 - \frac{\lambda}{n}\big) \to \lambda$.

(2) $S_n \sim \mathrm{Binomial}(n, \lambda/n)$*, so* $\mathbb{P}(S_n = k) = \dbinom{n}{k}\left(\dfrac{\lambda}{n}\right)^k\left(1 - \dfrac{\lambda}{n}\right)^{n-k}$.

(3) Write

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k} = \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\to 1} \cdot \frac{\lambda^k}{k!} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\to 1}.$$

For fixed k each labeled factor converges as indicated, so the product tends to $\dfrac{e^{-\lambda}\lambda^k}{k!}$.

(4) Thus $S_n \approx \mathrm{Poisson}(\lambda)$*. For* $\lambda = 2$*:* $\mathbb{P}(S_n = 0) \approx e^{-2} \approx 0.1353$ *and* $\mathbb{P}(S_n \geq 1) \approx 1 - e^{-2} \approx 0.8647$.

Problem 5. Let $X \sim \mathrm{Uniform}(0, 1)$ and fix an integer $n \geq 1$. Define $Y = X^n$.

(1) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.
(2) Find the probability density function of Y.
(3) Compute $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

Solution. *(1)* $\mathbb{E}[X] = \frac{1}{2}$ *and* $\mathrm{Var}(X) = \frac{1}{12}$.

(2) For $0 < y < 1$*,* $F_Y(y) = P(X^n \leq y) = P(X \leq y^{1/n}) = y^{1/n}$*. Differentiating,*

$$f_Y(y) = \frac{1}{n}\, y^{\frac{1}{n}-1}, \qquad 0 < y < 1,$$

and $f_Y(y) = 0$ *otherwise. (This is a* $\mathrm{Beta}(1/n, 1)$ *density.)*

(3) Directly, $\mathbb{E}[Y] = \mathbb{E}[X^n] = \displaystyle\int_0^1 x^n\, dx = \frac{1}{n+1}$*, and* $\mathbb{E}[Y^2] = \mathbb{E}[X^{2n}] = \dfrac{1}{2n+1}$*, so*

$$\mathrm{Var}(Y) = \frac{1}{2n+1} - \frac{1}{(n+1)^2}.$$

Solution. *(1) For* $0 < t < 1$*:* $F_M(t) = P(\text{all } \leq t) = t^3$*, so* $f_M(t) = 3t^2$*. And* $P(m > t) = (1-t)^3$*, so* $F_m(t) = 1 - (1-t)^3$ *and* $f_m(t) = 3(1-t)^2$.

(2) $\mathbb{E}[M] = \int_0^1 t \cdot 3t^2\, dt = \dfrac{3}{4}$*;* $\mathbb{E}[m] = \int_0^1 t \cdot 3(1-t)^2\, dt = \dfrac{1}{4}$.

(3) The k-th order statistic of n i.i.d. Uniform$(0,1)$ variables has density $\dfrac{n!}{(k-1)!(n-k)!}\, t^{k-1}(1-t)^{n-k}$*. For* $n = 3,\ k = 2$*,*

$$f_{X_{(2)}}(t) = \frac{3!}{1!\,1!}\, t(1-t) = 6\, t(1-t), \qquad 0 < t < 1,$$

a $\mathrm{Beta}(2,2)$ *density, symmetric about* $\frac{1}{2}$*, so* $\mathbb{E}[X_{(2)}] = \frac{1}{2}$.

(4) $\{x_1 + x_2 + x_3 \le 1,\ x_i \ge 0\}$ *is a simplex of volume* $\dfrac{1}{3!} = \dfrac{1}{6}$*, and the joint density on the unit cube is* 1*, so* $\mathbb{P}(X_1 + X_2 + X_3 \le 1) = \dfrac{1}{6}$.

Problem 6. (How many uniforms to exceed 1)

Let $U_1, U_2, \ldots$ be independent $\mathrm{Uniform}(0,1)$ random variables, and let

$$N = \min\{\, n \ge 1 :\ U_1 + U_2 + \cdots + U_n > 1 \,\}$$

be the first time the running sum exceeds 1.

(1) Show that for every integer $n \ge 0$,

$$\mathbb{P}(N > n) = \mathbb{P}(U_1 + \cdots + U_n \le 1) = \frac{1}{n!}.$$

(For $n = 3$ this is part (4) of Problem 6; the general case is the volume of an n-dimensional simplex. Take $\mathbb{P}(N > 0) = 1$.)

(2) Using $\mathbb{E}[N] = \sum_{n \ge 0} \mathbb{P}(N > n)$ for a nonnegative integer-valued random variable, show that $\mathbb{E}[N] = e$.

(3) Simulate the experiment by computer to confirm your answer. What is the smallest number of trials you needed to match $\mathbb{E}[N] = e$ to within 10^{-2}? To within 10^{-3}? You may use any software package; include your code in your write-up.

Solution. *(1) The event* $\{N > n\}$ *means the first* n *uniforms have not yet exceeded* 1*, i.e.* $U_1 + \cdots + U_n \le 1$*. The region* $\{u_1 + \cdots + u_n \le 1,\ u_i \ge 0\}$ *is the standard* n*-simplex, of volume* $1/n!$*, and the joint density of* $(U_1, \ldots, U_n)$ *is* 1 *on the unit cube, so* $\mathbb{P}(N > n) = 1/n!$ *(with the convention* $\mathbb{P}(N > 0) = 1 = 1/0!$*).*

(2) For a nonnegative integer-valued random variable, $\mathbb{E}[N] = \sum_{n \ge 0} \mathbb{P}(N > n)$*. Hence*

$$\mathbb{E}[N] = \sum_{n \ge 0} \frac{1}{n!} = e \approx 2.71828.$$

(3) A Monte Carlo simulation (repeatedly adding uniforms until the sum exceeds 1*, recording the count, and averaging) gives a sample mean converging to* e*. As the estimate is an average of independent trials, its typical error scales like* $N^{-1/2}$*: matching* e *to within* 10^{-2} *requires on the order of* 10^3–10^4 *trials, and to within* 10^{-3} *roughly two orders of magnitude more (on the order of* 10^5–10^6*).*