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- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. Two identical-looking bags sit on a table. Bag 1 contains 3 red and 7 green balls; Bag 2 contains 6 red and 4 green balls. You pick one of the two bags at random (each with probability $1/2$) and then draw balls from that same bag *without replacement*. The first two balls you draw are both red.

- (a) What is the probability that you are drawing from Bag 1?
- (b) Given the two red balls already observed, what is the probability that the *next* (third) ball drawn from the same bag is also red?

Solution. Let B_1, B_2 be the events of having chosen each bag, and let RR denote drawing two reds on the first two draws.

(a) The likelihoods are

$$P(RR | B_1) = \frac{3}{10} \cdot \frac{2}{9} = \frac{1}{15}, \quad P(RR | B_2) = \frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}.$$

By Bayes' rule,

$$P(B_1 | RR) = \frac{\frac{1}{2} \cdot \frac{1}{15}}{\frac{1}{2} \cdot \frac{1}{15} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{1/15}{1/15 + 1/3} = \frac{1/15}{6/15} = \frac{1}{6}, \quad P(B_2 | RR) = \frac{5}{6}.$$

(b) After removing two reds, Bag 1 holds 1 red and 7 green (so $P = \frac{1}{8}$) and Bag 2 holds 4 red and 4 green (so $P = \frac{1}{2}$). Averaging over the posterior from part (a),

$$P(3rd \text{ red} | RR) = \frac{1}{6} \cdot \frac{1}{8} + \frac{5}{6} \cdot \frac{1}{2} = \frac{1}{48} + \frac{20}{48} = \frac{21}{48} = \frac{7}{16} \approx 0.4375.$$

Problem 2. Urn A contains 2 white and 3 black balls; urn B contains 3 white and 5 black balls. One ball is drawn at random from A and transferred (without looking at its color) into B . A ball is then drawn at random from B .

- (a) What is the probability that the ball drawn from B is white?
- (b) Given that the ball drawn from B is white, what is the probability that the transferred ball was white?
- (c) Let T be the event "the transferred ball is white" and W the event "the ball drawn from B is white". Are T and W independent?

Solution. Initially $P(T) = \frac{2}{5}$ and $P(T^c) = \frac{3}{5}$. After the transfer, urn B holds 9 balls: 4 white if T occurred, 3 white if T^c occurred.

(a) By total probability,

$$P(W) = P(W | T)P(T) + P(W | T^c)P(T^c) = \frac{4}{9} \cdot \frac{2}{5} + \frac{3}{9} \cdot \frac{3}{5} = \frac{8}{45} + \frac{9}{45} = \frac{17}{45} \approx 0.3778.$$

(b) By Bayes' rule,

$$P(T | W) = \frac{P(W | T)P(T)}{P(W)} = \frac{8/45}{17/45} = \frac{8}{17} \approx 0.4706.$$

(c) $P(TW) = \frac{8}{45} \approx 0.1778$, while $P(T)P(W) = \frac{2}{5} \cdot \frac{17}{45} = \frac{34}{225} \approx 0.1511$. Since these are unequal, T and W are not independent.

Problem 3. A box contains three cards. One card is red on both sides (R/R), one is black on both sides (B/B), and one is red on one side and black on the other (R/B). A card is chosen at random and laid on the table; the side that shows is red.

(a) Given that the visible side is red, what is the probability that the hidden side is also red?

(b) A common but incorrect answer to (a) is $1/2$ ("the card is either R/R or R/B, so it is equally likely to be either"). Explain precisely what is wrong with this reasoning.

(c) Now suppose the box instead contains *two* R/B cards and *one* R/R card (still three cards). Again a card is drawn and the visible side is red. What is the probability that the hidden side is red?

Solution. Form the sample space of the face that is observed (not the card) since every face is equally likely to be the one showing.

(a) There are three red faces in the box: two on the R/R card and one on the R/B card, all equally likely to be the visible red face. The hidden side is red exactly when the showing face is one of the two on the R/R card. Hence

$$P(\text{hidden red} | \text{visible red}) = \frac{2}{3}.$$

(b) The two cards R/R and R/B are not equally likely given a red face. Seeing red is twice as likely to come from R/R (which has two red faces) as from R/B (which has one). Conditioning must weight each card by how likely it was to produce the observed red face; ignoring that weighting is the error. Formally, $P(R/R | \text{red}) = \frac{1 \cdot 1}{1 \cdot 1 + \frac{1}{2} \cdot 1} = 2/3$, not $1/2$.

(c) Now the red faces are: two on the single R/R card and one on each of the two R/B cards, for $2 + 2 = 4$ equally likely red faces. The hidden side is red only for the two faces of the R/R card, so

$$P(\text{hidden red} | \text{visible red}) = \frac{2}{4} = \frac{1}{2}.$$

The answer genuinely depends on the composition of the box; here it really is $1/2$.

Problem 4. An urn contains 6 white and 4 black balls. Three balls are drawn one at a time, without replacement.

(a) What is the probability that the *second* ball drawn is black?

(b) What is the probability that the first two balls are white and the third is black?

(c) Given that the third ball drawn is black, what is the probability that the first two balls are both white?

(d) Suppose instead the three draws are made *with* replacement. Recompute the probabilities in parts (a) and (c) under this model, and compare.

Solution. (a) By symmetry, each of the 10 balls is equally likely to be in the second position, so the second ball is black with probability $\frac{4}{10} = \frac{2}{5}$. (One may also compute $\frac{6}{10} \cdot \frac{4}{9} + \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{5}$.)

$$(b) P(W,W,B) = \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{120}{720} = \frac{1}{6}.$$

(c) Using (a) and (b),

$$P(\text{first two white} | \text{third black}) = \frac{P(W,W,B)}{P(\text{third black})} = \frac{1/6}{2/5} = \frac{5}{12} \approx 0.4167.$$

(d) With replacement the draws are independent and each is white with probability $\frac{6}{10} = \frac{3}{5}$, black with probability $\frac{2}{5}$. Part (a): the second ball is black with probability $\frac{2}{5}$ (unchanged). Part (c): independence makes the third draw irrelevant to the first two, so $P(\text{first two white} \mid \text{third black}) = P(\text{first two white}) = (\frac{3}{5})^2 = \frac{9}{25} = 0.36$. Without replacement the first two draws are negatively correlated with a later black, which is why the conditional probability $5/12 \approx 0.417$ exceeds the with-replacement value 0.36.

Problem 5. A bit $X \in \{0, 1\}$ with $P(X = 1) = 0.4$ is sent through a relay consisting of two binary symmetric channels in series. Each channel, independently, flips the bit passing through it with probability $\varepsilon = 0.2$. Let Y be the output of the first channel (the relay's intermediate value) and Z the final output.

- (a) Find the end-to-end error probability $P(Z \neq X)$.
- (b) Given that the receiver observes $Z = 1$, find the posterior probability $P(X = 1 \mid Z = 1)$.
- (c) Are X and Z conditionally independent given Y ? Justify your answer.

Solution. (a) An end-to-end error occurs iff exactly one of the two independent channels flips the bit:

$$P(Z \neq X) = 2\varepsilon(1 - \varepsilon) = 2(0.2)(0.8) = 0.32.$$

(If both flip or neither flips, the bit is delivered correctly.) So the relay behaves like a single binary symmetric channel with crossover probability $q = 0.32$.

- (b) With $P(Z = 1 \mid X = 1) = 1 - q = 0.68$ and $P(Z = 1 \mid X = 0) = q = 0.32$,

$$P(Z = 1) = 0.68(0.4) + 0.32(0.6) = 0.272 + 0.192 = 0.464,$$

$$P(X = 1 \mid Z = 1) = \frac{0.68(0.4)}{0.464} = \frac{0.272}{0.464} = \frac{17}{29} \approx 0.5862.$$

(c) Yes. Given the intermediate value Y , the output Z is determined solely by Y and the second channel's noise, which is generated independently of X . Hence $P(Z \mid X, Y) = P(Z \mid Y)$, i.e., X and Z are conditionally independent given Y . (They are not unconditionally independent, since Z carries information about X through Y .)

Problem 6. A short quiz has three questions, worth 1, 2 and 3 points. A student answers them correctly, independently of one another, with probabilities 0.8, 0.5 and 0.3 respectively. Let X be the student's total score.

- (a) Find the range of X and compute the pmf $p_X(x)$, indicating which outcomes of the experiment contribute to each value. Verify that $\sum_x p_X(x) = 1$.
- (b) Passing requires at least 3 points. Find $P(X \geq 3)$.
- (c) Write the CDF $F_X(x)$ for all real x , and use it to compute $P(1 < X \leq 4)$.

Solution. Write an outcome as (a_1, a_2, a_3) , where $a_i = 1$ if question i is answered correctly and $a_i = 0$ otherwise. The a_i are independent with $P(a_1=1) = 0.8$, $P(a_2=1) = 0.5$, $P(a_3=1) = 0.3$, and $X = a_1 + 2a_2 + 3a_3$.

- (a) There are $2^3 = 8$ outcomes; each has probability equal to the product of the three relevant factors:

(a_1, a_2, a_3)	000	100	010	110	001	101	011	111
probability	.07	.28	.07	.28	.03	.12	.03	.12
X	0	1	2	3	3	4	5	6

So $\text{Range}(X) = \{0, 1, 2, 3, 4, 5, 6\}$. Every value comes from a single outcome except $X = 3$, which is produced both by 110 (questions 1 and 2) and by 001 (question 3 alone); those two probabilities must be added:

x	0	1	2	3	4	5	6
$p_X(x)$	0.07	0.28	0.07	0.31	0.12	0.03	0.12

and $0.07 + 0.28 + 0.07 + 0.31 + 0.12 + 0.03 + 0.12 = 1$, as required.

(b) $P(X \geq 3) = 0.31 + 0.12 + 0.03 + 0.12 = 0.58$.

(c) F_X is the step function that accumulates these masses:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.07, & 0 \leq x < 1, \\ 0.35, & 1 \leq x < 2, \\ 0.42, & 2 \leq x < 3, \\ 0.73, & 3 \leq x < 4, \\ 0.85, & 4 \leq x < 5, \\ 0.88, & 5 \leq x < 6, \\ 1, & x \geq 6. \end{cases}$$

Hence $P(1 < X \leq 4) = F_X(4) - F_X(1) = 0.85 - 0.35 = 0.50$, which agrees with $p_X(2) + p_X(3) + p_X(4) = 0.07 + 0.31 + 0.12$.

Problem 7. Four data bits are encoded into a block of $n = 7$ bits and these 7 bits are sent over a noisy link one by one. Each of the 7 bits is flipped, independently of the others, with probability ε . The encoding is constructed in a way that allows to recover all four data bits if and only if at most one bit of the block of 7 was flipped. Let X be the number of flipped bits in the block.

(a) Which random variable does X represent and what is the distribution of X ? Write down $p_X(k)$ and justify it.

(b) Take $\varepsilon = 0.1$ and compute $P(X = 0)$, $P(X = 1)$ and $P(X \geq 2)$.

(c) Still at $\varepsilon = 0.1$, find the probability that the four data bits are recovered, and compare it with sending the four data bits over the same channel with no coding at all — which succeeds only if none of those four bits is flipped.

(d) Show that for every ε the ratio of the two success probabilities in (c) equals $(1 - \varepsilon)^2(1 + 6\varepsilon)$. Deduce that the code beats no coding for every $0 < \varepsilon < \frac{1}{2}$, with equality at $\varepsilon = \frac{1}{2}$, and explain what is special about $\varepsilon = \frac{1}{2}$.

(e) Simulate the transmission using the following encoding: the data bits x_1, x_2, x_3, x_4 are transmitted as a 7-bit block $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, where

$$x_5 = x_1 \oplus x_2 \oplus x_4, \quad x_6 = x_1 \oplus x_3 \oplus x_4, \quad x_7 = x_2 \oplus x_3 \oplus x_4$$

and where $\oplus$ is the XOR operation. Your task is to approximate the probability of incorrect decoding, i.e., the probability of the event that the string received on the output of the communication link differs from $\mathbf{x}$ by more than one bit. Perform 10^2 ; 10^3 ; 10^5 simulation rounds and record the probability values. How many rounds did you need to approximate the exact value of the probability of failed data recovery from the encoded block to within 10^{-3} ?

Solution. (a) The 7 bits are flipped independently, each with the same probability ε , so X counts the successes in 7 Bernoulli(ε) trials:

$$X \sim \text{Bin}(7, \varepsilon), \quad p_X(k) = \binom{7}{k} \varepsilon^k (1 - \varepsilon)^{7-k}, \quad k = 0, 1, \dots, 7.$$

(b) At $\varepsilon = 0.1$,

$$\begin{aligned} P(X = 0) &= (0.9)^7 = 0.4783, & P(X = 1) &= 7(0.1)(0.9)^6 = 0.3720, \\ P(X \geq 2) &= 1 - P(X \leq 1) = 1 - 0.8503 = 0.1497. \end{aligned}$$

(c) The decoder succeeds iff $X \leq 1$, so the coded success probability is

$$P(X \leq 1) = (0.9)^7 + 7(0.1)(0.9)^6 = \frac{531441}{625000} = 0.8503,$$

against $(0.9)^4 = 0.6561$ with no coding. The three extra transmitted bits raise the chance of delivering the four data bits from about 66% to about 85%.

(d) In general

$$P(X \leq 1) = (1 - \varepsilon)^7 + 7\varepsilon(1 - \varepsilon)^6 = (1 - \varepsilon)^6(1 + 6\varepsilon), \quad P(\text{uncoded success}) = (1 - \varepsilon)^4,$$

so the ratio is $(1 - \varepsilon)^2(1 + 6\varepsilon)$, as claimed. Expanding,

$$(1 - \varepsilon)^2(1 + 6\varepsilon) - 1 = 4\varepsilon - 11\varepsilon^2 + 6\varepsilon^3 = \varepsilon(6\varepsilon^2 - 11\varepsilon + 4) = 6\varepsilon\left(\varepsilon - \frac{1}{2}\right)\left(\varepsilon - \frac{4}{3}\right).$$

For $0 < \varepsilon < \frac{1}{2}$ both bracketed factors are negative, so the product is positive and the ratio exceeds 1: the code wins. At $\varepsilon = \frac{1}{2}$ the ratio is exactly 1, and indeed both schemes then succeed with probability $(\frac{1}{2})^4 = \frac{1}{16}$: when $\varepsilon = \frac{1}{2}$ the output of the channel is independent of its input, the channel carries no information whatever, and both schemes are reduced to guessing the four bits. No code can help there.