

ENEE324, Home assignment 1. Date due September 13 2026, 11:59pm .

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Please upload your work as a **single PDF file** to ELMS (under the “Assignments” tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. A hexadecimal device ID consists of 5 characters, each drawn from the 16 hex digits $\{0, 1, \dots, 9, A, B, C, D, E, F\}$.

- (a) How many device IDs are possible if repetition is allowed?
- (b) How many device IDs are possible if no character may repeat?
- (c) How many device IDs begin with a letter (one of A–F) and end with an even hex digit (one of 0, 2, 4, 6, 8, A, C, E), with the remaining three positions unrestricted (repetition allowed)?

Solution. *This is the product rule with 16 choices per unrestricted position.*

- (a) Each of the 5 positions is filled independently: $16^5 = 1,048,576$.
- (b) Choose the characters left to right without reuse: $16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 = 524,160$.
- (c) The first position has 6 choices (A–F), the last has 8 choices (the even hex digits), and each of the 3 middle positions has 16 choices:

$$6 \cdot 16^3 \cdot 8 = 6 \cdot 4096 \cdot 8 = 196,608.$$

Problem 2. A parts drawer contains 7 red, 5 blue, and 3 yellow resistors. Three resistors are drawn sequentially without replacement.

- (a) How many ordered outcomes are possible? (*ordered* means the sequence R,B,Y is different from B,R,Y)
- (b) How many ordered outcomes consist of three resistors of three different colors?
- (c) What is the probability that at least one of the three resistors drawn is yellow?

Solution. *There are $7 + 5 + 3 = 15$ resistors.*

- (a) Ordered draws of 3 from 15 without replacement: $15 \cdot 14 \cdot 13 = 2730$.
- (b) Choose one resistor of each color ($7 \cdot 5 \cdot 3 = 105$ ways) and then order the three positions ($3! = 6$ ways): $105 \cdot 6 = 630$.
- (c) Work with unordered subsets (the probability is the same either way). The number of ways to draw 3 resistors with no yellow is $\binom{12}{3} = 220$, out of $\binom{15}{3} = 455$ total, so

$$P(\text{no yellow}) = \frac{220}{455} = \frac{44}{91}, \quad P(\text{at least one yellow}) = 1 - \frac{44}{91} = \frac{47}{91} \approx 0.5165.$$

Problem 3. You distribute 12 identical software licenses among 5 distinct departments.

- (a) How many possible distributions are there?
- (b) How many distributions are there if each department receives at least one license?
- (c) How many distributions are there if no department receives more than 4 licenses?

Solution. Distributing n identical items into k distinct boxes has $\binom{n+k-1}{k-1}$ solutions (“stars and bars”).

(a) $\binom{12+5-1}{5-1} = \binom{16}{4} = 1820$.

(b) Give each department one license first, then distribute the remaining $12 - 5 = 7$ freely: $\binom{7+5-1}{5-1} = \binom{11}{4} = 330$.

(c) Count solutions of $x_1 + \cdots + x_5 = 12$ with each $x_i \leq 4$ by inclusion–exclusion. Let A_i be the event $x_i \geq 5$. Substituting $y_i = x_i - 5$ (sum 7) gives $\binom{11}{4} = 330$ for each single violation, and for two simultaneous violations (sum 2) gives $\binom{6}{4} = 15$; three violations are impossible. Hence

$$\binom{16}{4} - \binom{5}{1}\binom{11}{4} + \binom{5}{2}\binom{6}{4} = 1820 - 1650 + 150 = 320.$$

Problem 4. A project team of 4 engineers is chosen from a pool of 7 hardware engineers and 5 software engineers.

- (a) How many teams are possible?
- (b) How many teams contain at least one software engineer?
- (c) What is the probability that a randomly chosen team consists of exactly 2 hardware engineers and 2 software engineers?

Solution. The pool has $7 + 5 = 12$ engineers.

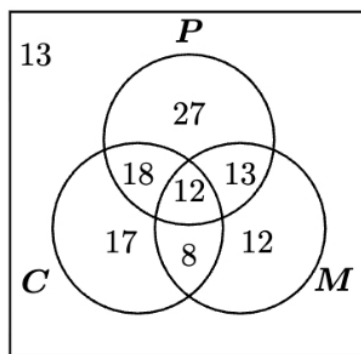
(a) $\binom{12}{4} = 495$.

(b) Complement: teams with no software engineer are $\binom{7}{4} = 35$, so $495 - 35 = 460$.

(c) $P = \frac{\binom{7}{2}\binom{5}{2}}{\binom{12}{4}} = \frac{21 \cdot 10}{495} = \frac{210}{495} = \frac{14}{33} \approx 0.4242$.

Problem 5. In a class of 120 students, 70 know Python, 55 know C, and 45 know MATLAB. Moreover, 30 know both Python and C, 25 know both Python and MATLAB, 20 know both C and MATLAB, and 12 know all three languages.

- (1) Draw a Venn diagram of the three sets and label all eight regions with their numerical values.
- (2) How many students know none of the three languages?
- (3) How many students know exactly one of the three languages?
- (4) How many students know Python but neither C nor MATLAB?
- (5) Express, in set notation, the set of students who know at least two of the three languages, and give its size.



Solution. Write P, C, M for the three sets, with $|P| = 70$, $|C| = 55$, $|M| = 45$, pairwise intersections 30, 25, 20, and $|P \cap C \cap M| = 12$.

(1) Work outward from the center. Center = 12; the “exactly two” regions are $|P \cap C| - 12 = 18$, $|P \cap M| - 12 = 13$, $|C \cap M| - 12 = 8$; the “exactly one” regions are

P only = $70 - (18 + 13 + 12) = 27$, C only = $55 - (18 + 8 + 12) = 17$, M only = $45 - (13 + 8 + 12) = 12$.

(2) By inclusion–exclusion, $|P \cup C \cup M| = 70 + 55 + 45 - 30 - 25 - 20 + 12 = 107$, so $120 - 107 = 13$ students know none.

(3) Exactly one = $27 + 17 + 12 = 56$ (equivalently $170 - 2(75) + 3(12) = 56$).

(4) Python only = 27.

(5) “At least two” = $(P \cap C) \cup (P \cap M) \cup (C \cap M)$. Its size is the three “exactly two” regions plus the center: $18 + 13 + 8 + 12 = 51$.

Problem 6. For each comparison below, fill in the blank with $=$, $<$, or $>$, and briefly justify your answer.

(a) $\binom{9}{4}$ ---- $\binom{9}{5}$

(b) # (distinct arrangements of the letters of “LEVEL”) ---- # (distinct arrangements of the letters of “LEVER”)

(c) $P(\text{sum of 6 with two fair dice})$ ---- $P(\text{sum of 9 with two fair dice})$

(d) $P(\text{at least one head in 3 fair flips})$ ---- $P(\text{exactly two heads in 3 fair flips})$

Solution. (a) $=$. By symmetry $\binom{9}{4} = \binom{9}{5} = 126$.

(b) $<$. “LEVEL” has two L’s and two E’s: $\frac{5!}{2!2!} = 30$. “LEVER” has only two E’s repeated: $\frac{5!}{2!} = 60$. So $30 < 60$.

(c) $>$. Sum 6 occurs in 5 ways (1+5, 2+4, 3+3, 4+2, 5+1) and sum 9 in 4 ways (3+6, 4+5, 5+4, 6+3), so $\frac{5}{36} > \frac{4}{36}$.

(d) $>$. $P(\text{at least one head}) = 1 - \frac{1}{8} = \frac{7}{8}$, while $P(\text{exactly two heads}) = \binom{3}{2}/8 = \frac{3}{8}$.

Problem 7 (Monte Carlo simulation). This problem asks you to *estimate* probabilities by simulation and compare them to the exact values. Use Python with NumPy (or an equivalent tool). Submit your code, your numerical estimates, and a short discussion.

(a) Write a simulation that reproduces the experiment of Problem 2: draw 3 resistors without replacement from a drawer of 7 red, 5 blue, and 3 yellow. Using $N = 10^6$ trials, estimate $P(\text{at least one yellow})$ and compare to the exact value $47/91$.

(b) *Birthday problem.* In a room of 25 people with birthdays independent and uniform over 365 days (ignore leap years), estimate $P(\text{at least three people share a birthday})$ using $N = 10^6$ trials, and compare to

the exact value

$$1 - \frac{1}{365^{25}} \sum_{j=0}^{12} \binom{365}{j} \binom{365-j}{25-2j} \frac{25!}{2^j},$$

in which the term indexed by j counts the outcomes where exactly j birthdays are shared by two people each and the remaining $25 - 2j$ people have birthdays of their own.

(c) For part (b), run the simulation with $N = 10^3, 10^4, 10^5, 10^6$ and tabulate the estimate against N . Comment on how the error behaves as N grows.

Solution. A reference implementation:

```
import numpy as np
from math import comb, factorial

rng = np.random.default_rng(0)

# (a) resistors: 7 red, 5 blue, 3 yellow -> code yellow as 2
drawer = np.array([0]*7 + [1]*5 + [2]*3) # 15 resistors
N = 10**6
hits = 0
for _ in range(N):
    draw = rng.choice(drawer, size=3, replace=False)
    hits += (2 in draw)
print("a) estimate:", hits/N, " exact:", 1 - comb(12,3)/comb(15,3))

# (b) birthday problem, 25 people: is some day a birthday of 3 or more?
def triple_estimate(N, n=25, days=365, batch=10**5):
    hits = 0
    for start in range(0, N, batch): # batch trials to limit memory
        m = min(batch, N - start)
        b = np.sort(rng.integers(0, days, size=(m, n)), axis=1)
        # a sorted row repeats a day 3+ times iff b[k+2] == b[k]
        hits += int((b[:, 2:] == b[:, :-2]).any(axis=1).sum())
    return hits / N

# complement of (b): every day is used at most twice
def exact_triple(n=25, days=365):
    tot = sum(comb(days, j) * comb(days-j, n-2*j) * factorial(n) // 2**j
               for j in range(n//2 + 1)) # j = number of doubled days
    return 1 - tot / days**n

print("b) estimate:", triple_estimate(10**6), " exact:", exact_triple())

# (c) convergence
for N in [10**3, 10**4, 10**5, 10**6]:
    print(f"N={N:>8}: {triple_estimate(N):.4f}")
```

(a) The exact value is $1 - \binom{12}{3} / \binom{15}{3} = 47/91 \approx 0.5165$. A run of $N = 10^6$ trials gives an estimate close to 0.5165 (typically within ± 0.001).

(b) The event fails precisely when no day is used more than twice. If exactly j days are used twice, the other $25 - 2j$ people occupy distinct days of their own, so the number of favorable assignments is $\binom{365}{j} \binom{365-j}{25-2j} 25! / 2^j$: choose the doubled days, choose the single days, order the 25 people, and divide by $2!$

for each doubled day. Summing over j and passing to the complement,

$$P(\text{at least three share a birthday}) = 1 - \frac{1}{365^{25}} \sum_{j=0}^{12} \binom{365}{j} \binom{365-j}{25-2j} \frac{25!}{2^j} \approx 0.01641,$$

and the simulation returns a value near 0.0164. It is worth contrasting this with the familiar two-person question, whose answer for the same 25 people is $1 - \prod_{k=0}^{24} \frac{365-k}{365} \approx 0.5687$: a shared birthday is likely, while a triple is rare (one needs 88 people before a triple is more likely than not).

(c) A representative table:

N	estimate	$\sqrt{p(1-p)/N}$
10^3	0.010–0.019	0.0040
10^4	0.015–0.017	0.0013
10^5	≈ 0.0166	0.0004
10^6	≈ 0.0164	0.0001

The estimate is an average of N independent Bernoulli outcomes, so by the law of large numbers it converges to the true probability, and its standard deviation scales like $\sqrt{p(1-p)/N} \propto N^{-1/2}$: each tenfold increase in N cuts the typical error by about a factor of $\sqrt{10} \approx 3.16$. Since $p \approx 0.016$ is small, it is the relative error $\sqrt{(1-p)/(pN)}$ that limits the accuracy here: at $N = 10^3$ only some 16 trials out of 1000 are successes and the estimate is good to no better than about $\pm 25\%$, whereas $N = 10^6$ yields roughly three correct digits. As a rule of thumb, an event of probability p needs N of order $100/p$ before its estimate is worth quoting. (Exact numbers vary with the random seed.)